A Novel Non-linear PD Control Design for the DC-DC Buck Converter

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Abstract

In this paper we study the DC-DC buck converter and its control problem. We solve the ODE corresponding to this problem analytically and analyse when the full system and its averaged model coincide. Finally, we suggest a new control function and show that it behaves well.

2.1 Introduction

During the Study Group Mathematics with Industry held in Eindhoven, we worked on a challenge formulated by DNV-GL about so-called buck-converters. The question originates in energy production and the way energy is produced nowadays. It used to be true that energy was produced in central plants, however in more recent years, solar panels, wind mills etc have appeared, in publicly as well as privately owned.

A situation that can serve as an example to keep in the back of the mind is a self-contained unit that produces its own energy and is not connected to the electricity network. Think of a yacht or a house, see Figure 2.1. In the house, several appliances are in use like a refrigerator, television and a washing machine. However, these appliances are not constantly turned on, and hence, do not use electricity all of the time. This fact can lead to a problem since the appliances all need an input voltage of approximately 220-230 V (in Europe).

So, the aim is to keep the voltage in the network of the house constant. When one of the appliances, for example the television, is turned on, the load in the network

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increases which leads to a voltage increase in the network—by using Ohms law and the fact that the current in the network does not respond immediately. Since this increase in voltage can damage the appliances, this jump is problematic and the voltage needs to be restored to its original value as quickly as possible.

Figure 2.1: A house that is energy neutral, with solar panels and several appliances.

Another problem comes from the fact that solar panels do not produce energy with the desired constant output voltage of 220-230 V. To overcome these problems, so-called buck-converters are used. Such systems are designed to reduce the input voltage from the solar panels to the desired value in the network. Moreover, they aim to keep the voltage in the system constant when appliances are turned on and off. A basic example of such a system is given in Figure 2.2(top). The topology of this electric circuit is simple, yet it contains almost all the difficulties associated with the study of power electronic converters. On the left, the input voltage $v_{\text{in}}(t)$ and the corresponding solar panels are located and on the right the load $R(t)$ represents the appliances such as the television. The network in the house is located within the pink box in the figure. A switch to control the voltage coming from the solar panels is placed at the arrow, Figure 2.2(bottom). This switch can be turned on, leading to an input of voltage into the system. When it is turned off no voltage is delivered to the system in pink. The output voltage in the network in the house is denoted by $v_0(t)$. The aim is to keep $v_0(t)$ at a constant target value which we denote by v_0^*. In diagram form, this is given by:
The corresponding electric circuit can be described by the following set of ODE's:

$$
\begin{aligned}
L\frac{\mathrm{d}i}{\mathrm{d}t} &= \mathrm{sw}(t)v_{\text{in}}(t) - v_o; \\
C\frac{\mathrm{d}v_o}{\mathrm{d}t} &= -\frac{v_o}{R} + i,
\end{aligned}
\tag{2.1}
$$

where L (inductance) and C (conductance) are (positive) constants and $\mathrm{sw}(t)$ represents the switch and has either the value 0 or 1.

Once every time-interval of fixed length T, the decision can be made to flip the switch on or off. This sequence of choices can be modelled by the function $D(t)$, the

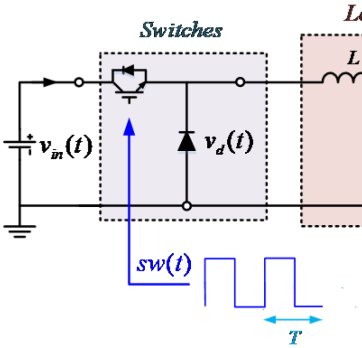

Figure 2.2: The electric circuit with the switch.

so-called duty ratio signal of the converter which lies in $[0, 1]$. At the start of every new time interval k of length T, the corresponding D_k is determined as $D_k = D(kT)$, see Figure 2.3.

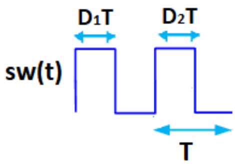

Figure 2.3: Sketch of the D_k.

2.1.1 Problem description

The goal of the converter is to keep its output voltage $v_o(t)$ constant at the desired level v_0^*. The control is carried out by a feedback mechanism monitoring $v_o(t)$ and adjusting the duty cycle ratio $D(t)$. The challenge is to test whether $v_o(t)$ decays back to v_o^* in a "nice" way for functions $D(v_o(t), t)$, in the case that $R(t)$ is a discontinuous function with jumps. More specifically, DNV-GL would like answers to several (related) questions:

1. Does $v_o(t)$ converge to v_o^*?
2. Does $v_o(t)$ converge fast to v_o^*?
3. Does $v_o(t)$ converge to v_o^* without too many oscillations?

Moreover, they would like an algorithm that generates an optimal $D(v_o(t), t)$. In our analysis, we made the following assumptions:

- The input voltage v_{in} is constant, although in practise this is a function of time;
- $R(t)$ is a step function: it changes at $t = 0$ from R_{old} to R_{new}.

Since the current does not respond immediately at $t = 0$, $v_o(t)$ does change to a new value which gives (by Ohm's law) that $v_o(0) = v_o^* \frac{R_{\text{new}}}{R_{\text{old}}}$ and $\frac{\mathrm{d}v_o}{\mathrm{d}t}(0) = 0$.

2.1.2 Approach

The problem can be separated into four different cases: the averaged system with and without feedback and the full system with and without feedback. Our approach for each of these cases is summarised in table 2.1.2.

In Section 2.2, we will introduce and study an averaged model. Averaging is the method which is used currently and the averaging method in a system without feedback is completely known. Also the averaging method in a system with feedback is already used, although not always exact. In table 2.1.2 a commonly used D is shown. For this $D = k_1(v_o^* - v_o) + k_2 \frac{\mathrm{d}v_o}{\mathrm{d}t} + k_3$ we performed a stability analysis which gives us for which choices of k_1, k_2 and k_3 the system is stable. The problem with the averaging method is that it is not always valid. By nondimensionalization we were able to show when the averaging method breaks down. Namely, when $\sqrt{LC} \sim T$.

Therefore we need to consider the full system, which we will do in Section 2.3. Since the full system is a second order inhomogeneous differential equation we can solve it analytically, which we will do in that section. We will also show that the solution converges for a constant D. This analytical solution is valid in both systems: with and without feedback. We implemented the analytical solution in MATLAB so it can be used for different functions D to determine whether the system converges to v_o^*.

A great advantage of using the analytical solution over numerical simulations is that it requires less computation time. The results will be explained in more detail in Section 2.4. In this section we also provide a new function D that makes the system converge faster than the functions D that are currently used.

	No Feedback: $D(t)$	Feedback: $D(v_o, t)$
Averaged system	Known	Stability analysis for $D = k_1(v_o^* - v_o) + k_2 \frac{\mathrm{d}v_o}{\mathrm{d}t} + k_3$
Full system	Analytical solution Convergence for constant D	Analytical solution + algorithm

2.2 Averaged model

In the case where the switching frequency is much higher than the natural frequency of the L-C system, we can average the system (2.1) and hence obtain a form in which we can analyse the stability of the system for various forms of the duty ratio, D.

To do this with some rigour, we must first non-dimensionalise the system. We non-dimensionalise in Section 2.2.1, then average in Section 2.2.2. This then allows us to explore stability in a very simple case, in Section 2.2.3.

2.2.1 Non-dimensionalisation

We non-dimensionalise the system (2.1) as follows:

$$\begin{aligned} v_o &= v_o^* \hat{v}_o; & i &= \frac{v_o^*}{R}\hat{i}; \\ t &= t^* \hat{t}; & v_{\text{in}} &= v_o^* \hat{v}_{\text{in}}, \end{aligned} \tag{2.2}$$

where t^* is a natural time-scale of the system to be determined. Note also that $\hat{v}_o^* = 1$. With this change of variables, (2.1) becomes:

$$\begin{aligned} \frac{L/R}{t^*}\frac{\mathrm{d}\hat{i}}{\mathrm{d}\hat{t}} &= \mathrm{sw}(t^*\hat{t})\hat{v}_{\text{in}}(t^*\hat{t}) - \hat{v}_o, \\ \frac{RC}{t^*}\frac{\mathrm{d}\hat{v}_o}{\mathrm{d}\hat{t}} &= -\hat{v}_o + \hat{i}. \end{aligned} \tag{2.3}$$

Equation (2.3) suggests choosing either $t^* = L/R$ or $t^* = RC$ as the natural time-scale. Instead, we choose the geometric mean of these two options, $t^* = \sqrt{LC}$. The reason for this choice is that, by combining the equations in (2.3) to obtain a single second-order ODE for v_o,

$$\frac{\sqrt{LC}}{(t^*)^2}\frac{\mathrm{d}^2\hat{v}_o}{\mathrm{d}\hat{t}^2} + \frac{L/R}{t^*}\frac{\mathrm{d}\hat{v}_o}{\mathrm{d}\hat{t}} + \hat{v}_o = \mathrm{sw}(t^*\hat{t})\hat{v}_{\text{in}}(t^*\hat{t}),$$

the time-scale that appears in front of the highest derivative is $t^* = \sqrt{LC}$. Further, $\sqrt{LC}$ is often quoted in the literature as the time-scale of an L-C circuit, for example by Mohan and Undeland (2007).

Then, letting $r = \sqrt{L/C}/R$, (2.3) becomes

$$\begin{aligned} r\frac{\mathrm{d}\hat{i}}{\mathrm{d}\hat{t}} &= \mathrm{sw}(t^*\hat{t})\hat{v}_{\text{in}}(t^*\hat{t}) - \hat{v}_o, \\ \frac{1}{r}\frac{\mathrm{d}\hat{v}_o}{\mathrm{d}\hat{t}} &= -\hat{v}_o + \hat{i}. \end{aligned} \tag{2.4}$$

The variables $\hat{v}_o$, $\hat{i}$ and $\hat{t}$ are all dimensionless, and usually $\mathcal{O}(1)$. Further, there are only two dimensionless parameters in (2.4): $r = \sqrt{L/C}/R$, and $\varepsilon = T/t^*$ (the latter being hidden inside $\mathrm{sw}(t^*\hat{t})$ – see Section 2.2.2). Ignoring ε for now, the fact that we have grouped R, C and L together into a single dimensionless parameter means that we can investigate the system while considering only one parameter; for example, there will be no change to the fundamental behaviour of the system if we double both L and C, since r will remain the same.

2.2.2 Averaged system

With the non-dimensionalisation of time-scale presented in (2.2), the switching function becomes

$$\begin{aligned} \mathrm{sw}(t^*\hat{t}) &= \begin{cases} 0 & \text{if } kT < t^*\hat{t} \le (k + D_{k+1})T, \\ 1 & \text{if } (k + D_{k+1})T < t^*\hat{t} \le (k+1)T \end{cases} \\ &= \begin{cases} 0 & \text{if } k < \hat{t}/\varepsilon \le (k + D_{k+1}), \\ 1 & \text{if } (k + D_{k+1}) < \hat{t}/\varepsilon \le (k+1) \end{cases} \\ &= \widehat{\mathrm{sw}}(\hat{t}/\varepsilon), \end{aligned}$$

where $\varepsilon = T/t^* = T/\sqrt{LC}$ is usually much less than one.

We now introduce the average

$$\bar{f}(\hat{t}) = \frac{1}{T}\int_{\hat{t}}^{\hat{t}+T} \hat{f}(s)\,\mathrm{d}s.$$

Since $\hat{v}_o$ and $\hat{i}$ (by the L-C equations) and $\hat{v}_{\text{in}}$ (by assumption) vary on the slow time-scale $\hat{t}$,

$$\bar{v}_o \approx \hat{v}_o, \quad \bar{i} \approx \hat{i}, \quad \bar{v}_{\text{in}} \approx \hat{v}_{\text{in}}.$$

However, $\widehat{\mathrm{sw}}(\hat{t}/\varepsilon)$ oscillates periodically on the fast time-scale $\hat{t}/\varepsilon$, and so

$$\overline{\mathrm{sw}}(\hat{t}/\varepsilon) = D(\hat{t}).$$

Hence, averaging (2.4) and letting $\bar{t} = \hat{t}$ for consistency of notation, we obtain

$$\begin{aligned} r\frac{\mathrm{d}\bar{i}}{\mathrm{d}\bar{t}} &= D\bar{v}_{\text{in}} - \bar{v}_o, \\ \frac{1}{r}\frac{\mathrm{d}\bar{v}_o}{\mathrm{d}\bar{t}} &= -\bar{v}_o + \bar{i}. \end{aligned} \tag{2.5}$$

Since we have only considered leading-order when averaging, the averaged form is accurate to $\mathcal{O}(\varepsilon)$. Hence, to obtain an accuracy of 10%, we need $\varepsilon < 0.1$, *i.e.* $\sqrt{LC} > 10T$.

As shown by Yue et al. (2012), one could average more rigorously by transforming to the fast-time scale $\hat{t}/\varepsilon$ and using alternative methods such as Krylov-Bogoliubov-Mitropolsky or Multi-Frequency-Averaging. This would then allow to consider higher-order terms, and increase the accuracy of the averaging. For example, if we include first-order terms as well as leading-order terms, the averaging is accurate to $\mathcal{O}(\varepsilon^2)$, instead of just $\mathcal{O}(\varepsilon)$. Hence an accuracy of 10% can be achieved with $\varepsilon \sim 0.3$ (so that $\varepsilon^2 \sim 0.1$), where before we needed $\varepsilon \sim 0.1$. Including progressively more higher-order terms allows to maintain high accuracy for values of ε that are closer to one (*i.e.* when $\sqrt{LC} \sim T$).

2.2.3 Stability analysis

Assuming $\sqrt{LC} \gg T$, we now analyse the stability of the system in the leading-order averaged case, (2.5). We consider the case without feedback, then explore how introducing feedback can improve the stability of the system.

Without feedback. When D is independent of $\bar{v}_o$ and $\bar{i}$, (2.5) can be written as the linear system

$$\frac{\mathrm{d}\bar{\mathbf{x}}}{\mathrm{d}\bar{t}} = \mathbf{A}\bar{\mathbf{x}} + \mathbf{b},$$

where

$$\bar{\mathbf{x}} = \begin{pmatrix} \bar{i} \\ \bar{v}_o \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & -1/r \\ r & -r \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} D\bar{v}_{\text{in}}/r \\ 0 \end{pmatrix}.$$

Note that this system has a steady state at $\bar{i} = \bar{v}_o = D\bar{v}_{\text{in}}$, so we should choose $D = \bar{v}_0^*/\bar{v}_{\text{in}} = 1/\bar{v}_{\text{in}}$.

The eigenvalues, λ, of the matrix $\mathbf{A}$ determine the stability of the system; here

$$\lambda = \frac{-r \pm \sqrt{r^2 - 4}}{2}.$$

Since $r > 0$, $\mathrm{Re}(\lambda) < 0$ for all r, and the system is unconditionally stable. However, if $r^2 < 4$, the eigenvalues are complex and hence the solution is oscillatory, which is not desired.

With feedback. We consider a simple PD-controller. In dimensional form, we have

$$\begin{aligned} D &= k_1(v_o - v_o^*) + k_2\frac{\mathrm{d}v_o}{\mathrm{d}t} + k_3 \\ &= k_1(v_o - v_o^*) + \frac{k_2}{C}\left(i - \frac{v_o}{R}\right) + k_3, \end{aligned} \tag{2.6}$$

where k_1, k_2 and k_3 are constants that we can choose to obtain fast convergence to a stable solution. Non-dimensionalising by (2.2) and averaging as in Section 2.2.2, (2.6) becomes

$$\begin{aligned} D &= k_1 v_o^* \left(\bar{v}_o - 1\right) + \frac{k_2 v_o^*}{RC}\left(\bar{i} - \bar{v}_o\right) + k_3 \\ &= \frac{1}{\bar{v}_{\text{in}}}\left(\bar{k}_1\left(\bar{v}_o - 1\right) + \bar{k}_2\left(\bar{i} - \bar{v}_o\right) + \bar{k}_3\right), \end{aligned}$$

where we have defined the dimensionless constants

$$\bar{k}_1 = k_1 v_o^* \bar{v}_{\text{in}}, \quad \bar{k}_2 = k_2 v_o^* \bar{v}_{\text{in}}/RC, \quad \bar{k}_3 = k_3 \bar{v}_{\text{in}}.$$

Now (2.5) can be written as the linear system

$$\frac{\mathrm{d}\bar{\mathbf{x}}_f}{\mathrm{d}\bar{t}} = \mathbf{A}_f\bar{\mathbf{x}}_f + \mathbf{b}_f,$$

where

$$\bar{\mathbf{x}}_f = \begin{pmatrix} \bar{i} \\ \bar{v}_o \end{pmatrix}, \quad \mathbf{A}_f = \begin{pmatrix} \bar{k}_2/r & (\bar{k}_1 - \bar{k}_2 - 1)/r \\ r & -r \end{pmatrix}, \quad \mathbf{b}_f = \begin{pmatrix} \left(-\bar{k}_1 + \bar{k}_3\right)/r \\ 0 \end{pmatrix}.$$

Choosing $\bar{k}_3 = 1$, then this system once again has a steady state at $\bar{i} = \bar{v}_o = 1$.

Now the eigenvalues of $\mathbf{A}_f$ are

$$\lambda_f = \frac{-(r - \bar{k}_2/r) \pm \sqrt{(r - \bar{k}_2/r)^2 + 4(\bar{k}_1 - 1)}}{2}.$$

We can now choose $\bar{k}_1$ and $\bar{k}_2$ in such a way that the system is stable and converges as quickly as possible.

Firstly, we choose $\bar{k}_2 < 0$ and $k_1 < 1$, so that $\mathrm{Re}(\lambda_f) < 0$ for all r and the system is stable. Secondly, in order to converge as quickly as possible without oscillations, both eigenvalues must be real, large and negative. Hence we aim to make $(r - \bar{k}_2/r)$ as large as possible, but also $(r - \bar{k}_2/r)^2 + 4(\bar{k}_1 - 1)$ positive and close to zero so that *both* eigenvalues are large and negative. Hence we choose $\bar{k}_1$ such that

$$1 - \frac{1}{4}\left(r - \frac{\bar{k}_2}{r}\right)^2 \lessapprox \bar{k}_1 < 1.$$

However, note that when choosing $\bar{k}_1$ and $\bar{k}_2$, we must be careful that D remains between 0 and 1, and varies on the slow time-scale $\bar{t}$ rather than the fast time-scale $\bar{t}/\varepsilon$ (otherwise averaging is not possible). We show an example of improved convergence using PD control in Section 2.4.

2.3 Analytic solution

In this section we explain how to find an analytical solution of the set of differential equations given in Equation (2.1):

$$L\frac{\mathrm{d}i}{\mathrm{d}t} = \mathrm{sw}(t)v_{\mathrm{in}} - v_o,$$
$$C\frac{\mathrm{d}v_o}{\mathrm{d}t} = -\frac{v_o}{R} + i,$$

with initial conditions $v_o(0) = v_o^* \frac{R_{\mathrm{new}}}{R_{\mathrm{old}}} =: v_c$ and $(v_o)_t(0) = 0$. This system of equations can be written as a second order differential equation given by

$$LC\frac{\mathrm{d}v_o}{\mathrm{d}t^2} + \frac{L}{R}\frac{\mathrm{d}v_o}{\mathrm{d}t} + v_o = v_{\mathrm{in}}\,\mathrm{sw}(t). \tag{2.7}$$

Since Equation (2.7) is a linear inhomogeneous second order differential equation it can be solved by using the method Variation of Parameters. Therefore, we first rewrite Equation (2.7) into standard form,

$$\frac{\mathrm{d}v_o}{\mathrm{d}t^2} + \frac{1}{RC}\frac{\mathrm{d}v_o}{\mathrm{d}t} + \frac{1}{LC}v_o = \frac{v_{\mathrm{in}}}{LC}\mathrm{sw}(t). \tag{2.8}$$

The first step is then to solve the corresponding homogeneous differential equation, which has two solutions $u_\pm(t)$, given by:

$$\begin{aligned} u_\pm &= e^{\lambda_\pm t}, \\ \lambda_\pm &:= -\frac{1}{2RC} \pm \frac{1}{2}\sqrt{\xi} \quad \xi := \frac{1}{R^2C^2} - \frac{4}{LC} \end{aligned} \tag{2.9}$$

Then according to the method Variation of Parameters, the general solution of the inhomogeneous Equation (2.7) is given by

$$v_o(t) = (A(t)+a)u_+(t) + (B(t)+b)u_-(t), \tag{2.10}$$

with $A(t)$ and $B(t)$ are given as:

$$\begin{aligned} A(t) &= -\int_0^t \frac{1}{W(x)} e^{\lambda_+ x} v_{\text{in}} \text{sw}(x)\, dx; \\ B(t) &= \int_0^t \frac{1}{W(x)} e^{\lambda_- x} v_{\text{in}} \text{sw}(x)\, dx, \end{aligned}$$

and W is the Wronskian given by

$$\begin{aligned} W(x) &= u_-(x)\dot{u}_+(x) - \dot{u}_-(x)u_+(x) = (\lambda_+ - \lambda_-)e^{(\lambda_+ + \lambda_-)x} \\ &= \sqrt{\xi}\, e^{-\frac{x}{RC}} \end{aligned}$$

Recall that $\text{sw}(x)$, the switching function, is given by:

$$\text{sw}(t = kT + t') = \begin{cases} 1 & \text{if } t' \in [0, D_{k+1}T] \\ 0 & \text{if } t' \in [D_{k+1}T, T] \end{cases}$$

By viewing the integrals in the expression of $A(t)$ and $B(t)$ as a sum of integrals with integration domain between iT and $(i+1)T$, where $i = 0, 1, \ldots$ and by using that $\frac{1}{RC} + \lambda_\pm = -\lambda_\mp$, and that $\text{sw}(x)$ is often zero, we can rewrite the expressions to:

$$A(t = kT + t') = -\frac{v_{\text{in}}}{\sqrt{\xi}} \left(\int_{kT}^{\max(t', D_{k+1}T)} e^{-\lambda_- x}\, dx + \sum_{i=1}^{k} \int_{(i-1)T}^{(i-1)T + D_i T} e^{-\lambda_- x}\, dx \right).$$

Therefore, we find that:

$$A(t = kT + t') =$$
$$\begin{cases} \frac{v_{\text{in}}}{\sqrt{\xi}\lambda_-} \left(e^{-\lambda_- t} - e^{-\lambda_- kT} + \sum_{i=1}^{k} e^{-\lambda_-(i-1)T} \left(e^{-\lambda_- D_i T} - 1 \right) \right) & \text{if } t' \in [0, D_{k+1}T]; \\ \frac{v_{\text{in}}}{\sqrt{\xi}\lambda_-} \sum_{i=1}^{k+1} e^{-\lambda_-(i-1)T} \left(e^{-\lambda_- D_i T} - 1 \right) & \text{if } t' \in [D_{k+1}T, T]; \end{cases}$$

$$B(t = kT + t') =$$
$$\begin{cases} -\frac{v_{\text{in}}}{\sqrt{\xi}\lambda_+} \left(e^{-\lambda_+ t} - e^{-\lambda_+ kT} + \sum_{i=1}^{k} e^{-\lambda_+(i-1)T} \left(e^{-\lambda_+ D_i T} - 1 \right) \right) & \text{if } t' \in [0, D_{k+1}T]; \\ -\frac{v_{\text{in}}}{\sqrt{\xi}\lambda_+} \sum_{i=1}^{k+1} e^{-\lambda_+(i-1)T} \left(e^{-\lambda_+ D_i T} - 1 \right) & \text{if } t' \in [D_{k+1}T, T]. \end{cases}$$

Using Equation (2.10), we find that the initial conditions are $v_o(0) = v_c$ and $(\mathrm{d}v_o/\mathrm{d}t)(0) = 0$, we are able to determine that a and b have to satisfy $a + b = v_c$ and $a\lambda_- + b\lambda_+ = 0$. Where a and b are the integration constants in the general solution Equation (2.10). It follows that:

$$a = -\frac{v_c\lambda_+}{\lambda_- - \lambda_+} = \frac{v_c\lambda_+}{\sqrt{\xi}}, \qquad b = \frac{v_c\lambda_-}{\lambda_- - \lambda_+} = -\frac{v_c\lambda_-}{\sqrt{\xi}},$$

where $\lambda_\pm$ and ξ are as before. Combining these results and simplifying the expressions gives the general solution of the differential equation 2.7, given by:

For $t \in [(k + D_{k+1})T, (k + 1)T]$**:**

$$\begin{aligned} v_o(t) = &\frac{\lambda_+}{\sqrt{\xi}} \left(v_c + LCv_{\text{in}} \sum_{i=1}^{k+1} e^{-\lambda_-(i-1)T} \left(e^{-\lambda_- D_i T} - 1\right) \right) e^{\lambda_- t} \\ &- \frac{\lambda_-}{\sqrt{\xi}} \left(v_c + LCv_{\text{in}} \sum_{i=1}^{k+1} e^{-\lambda_+(i-1)T} \left(e^{-\lambda_+ D_i T} - 1\right) \right) e^{\lambda_+ t} \end{aligned}$$

For $t \in [kT, (k + D_{k+1})T]$**:**

$$\begin{aligned} v_o(t) = &\frac{\lambda_+}{\sqrt{\xi}} \left(v_c + LCv_{\text{in}} \sum_{i=1}^{k} e^{-\lambda_-(i-1)T} \left(e^{-\lambda_- D_i T} - 1\right) \right) e^{\lambda_- t} \\ &- \frac{\lambda_-}{\sqrt{\xi}} \left(v_c + LCv_{\text{in}} \sum_{i=1}^{k} e^{-\lambda_+(i-1)T} \left(e^{-\lambda_+ D_i T} - 1\right) \right) e^{\lambda_+ t} \\ &- \frac{\lambda_+ v_{\text{in}}}{\sqrt{\xi}} e^{\lambda_-(t-kT)} + \frac{\lambda_- v_{\text{in}}}{\sqrt{\xi}} e^{\lambda_+(t-kT)} + v_{\text{in}} \end{aligned}$$

Note that the general solution still depends on the constants $\{D_k\}_k$. When applying this general solution one should determine a new D_{k+1} with the graph of the solution v_o until time kT. The choice of D_{k+1} then gives the graph and solution of v_o until time $(k + 1)T$.

2.3.1 Convergence for constant D

Like mentioned before, a priori we do not know the constants $\{D_k\}_k$ when we do have a feedback in our system. An example of a no feedback loop, where we do know the constants, is when all the D_k's are constant and equal to some $D \in [0, 1]$. This assumption simplifies the general solution and in this section we will show that the general solution converges to v_o^* for $t \to \infty$.

In the limit $t \to \infty$, we first note that $v_o(t)$ converges to a constant function if and only if the sequence $\{v_k := v_o(kT)\}_{k\in\mathbb{N}}$ converges to a constant. This allows us to simplify our computations. Note that:

$$\begin{aligned} v_k &= \frac{\lambda_+}{\sqrt{\xi}}\left(v_c e^{\lambda_- kT} + LCv_{\text{in}}(e^{-\lambda_- DT} - 1)e^{\lambda_- kT}\sum_{i=1}^{k} e^{-\lambda_-(i-1)T}\right) \\ &- \frac{\lambda_-}{\sqrt{\xi}}\left(v_c e^{\lambda_+ kT} + LCv_{\text{in}}(e^{-\lambda_+ DT} - 1)e^{\lambda_+ kT}\sum_{i=1}^{k} e^{-\lambda_+(i-1)T}\right) \\ &= \frac{\lambda_+}{\sqrt{\xi}}\left(v_c e^{\lambda_- kT} + LCv_{\text{in}}(e^{-\lambda_- DT} - 1)\frac{e^{\lambda_- kT} - 1}{1 - e^{-\lambda_- T}}\right) \\ &- \frac{\lambda_-}{\sqrt{\xi}}\left(v_c e^{\lambda_+ kT} + LCv_{\text{in}}(e^{-\lambda_+ DT} - 1)\frac{e^{\lambda_+ kT} - 1}{1 - e^{-\lambda_+ T}}\right) \end{aligned}$$

Since $e^{-x} \to 0$ if $x \to \infty$ and $\sqrt{\xi} < \frac{1}{RC}$ shows that $\lambda_\pm < 0$, we see that:

$$\lim_{k\to\infty} v_k = -\frac{\lambda_+ LCv_{\text{in}}}{\sqrt{\xi}(1 - e^{-\lambda_- T})}(e^{-\lambda_- DT} - 1) + \frac{\lambda_- LCv_{\text{in}}}{\sqrt{\xi}(1 - e^{-\lambda_+ T})}(e^{-\lambda_+ DT} - 1).$$

Therefore, v_k converges for $k \to \infty$ and hence so does $v_o(t)$ if $t \to \infty$. Finally, note that one can choose D to be such that $\lim_{t\to\infty} v_o(t) = v_o^*$ by solving the above equation.

2.4 Results

In this section, specific examples are given to demonstrate the theoretical work in this report. We will show that the analytical solution coincides with a numerical approximation of the system of ODEs. Then, we will present a buck converter where the characteristic timescale $\tau = \sqrt{LC}$ is comparable to the switching period T. In this case, one expects to see a significant difference between a solution of the real system and a solution of the averaged system. We will show that a PD controller designed for the averaged system is not able to stabilize the full system. To stabilize the system, we will use a (new) nonlinear feedback function. After tuning its control parameters, the system stabilizes within 10 switching periods.

Consider a buck converter with $L = 1.3$mH, $C = 81\mu$F and $T = 0.2$ms — so a switching frequency of $f = 5$kHz. The input voltage is assumed to be $v_{in} = 1000V$ constantly, with a reference output voltage of $v_o^* = 500$V. Hence, in a stabilized state the switching duty ratio D should be at 0.5. In our examples we suppose the following scenario. At some point in time the output voltage is stabilized at the reference 500V with a load resistance of $R = 5$ Ohm, thus with a steady state current of 100A. Suddenly the load resistance drops to 2 Ohm, e.g. after a new device is plugged into the power network. The current cannot change immediately, due to the inductor, and so the output voltage drops to 100A $\times$2Ohm $= 200$V. We take this as the starting point of our calculations, that is, we solve the system with the initial conditions $v_o(0) = 200, \dot{v}_o(0) = 0$.

In Figure 2.4 the analytical solution and a numerical approximation of the system are given. Here a fixed duty ratio of $D = 0.5$ is used. From the figure it is clear that the solutions coincide, numerically confirming the theory. Also, we see that the output voltage v_o slowly stabilizes around 500V.

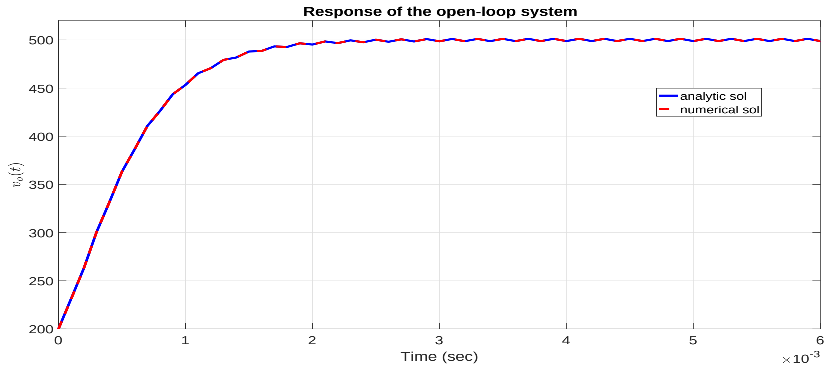

Figure 2.4: Comparison of the analytic solution (blue solid line) and the numerical solution (red dashed line) for the open-loop system.

In the given example, we have $\sqrt{LC} = 0.3245$ms which is comparable to $T = 0.2$ms. So according to the theory, the averaged system is an incorrect simplification of the full system. We will illustrate this by using a standard feedback function. For the averaged system, a standard PD controller is given by

$$D(t) = k_1(v_o(t) - v_o^*) + k_2 v_o'(t) + k_c.$$

The parameters k_1, k_2, k_c need to be chosen, where k_c is given by the steady state duty ratio, so equal to 0.5 in this case. The standard PD tuning method employs the Nyquist stability criterion and the Bode plots. If one wants to utilize the time-domain performance to tune those parameters, experiences and some trials and errors are needed. Figure 2.5 shows solutions of the averaged system, one solution is found without using any control — so a constant D —, and for the other solution a PD controller is applied. The PD controlled system has faster convergence, as desired.

Figure 2.6 shows the solution of the full system using the same PD controller. Inspecting the output voltage v_o, it appears as if the system is stabilized with a steady state within 5% of the reference output. However, the duty ratio is switching between 0 and 1 rather than converging to the constant ratio 0.5. This switching behaviour is not desirable for real-life systems, and consequently, this standard PD controller is not acceptable.

From theoretical point of view, we already knew that the averaged system is a poor model for the full system if $\sqrt{LC} \sim T$. To solve the issue, we propose using the

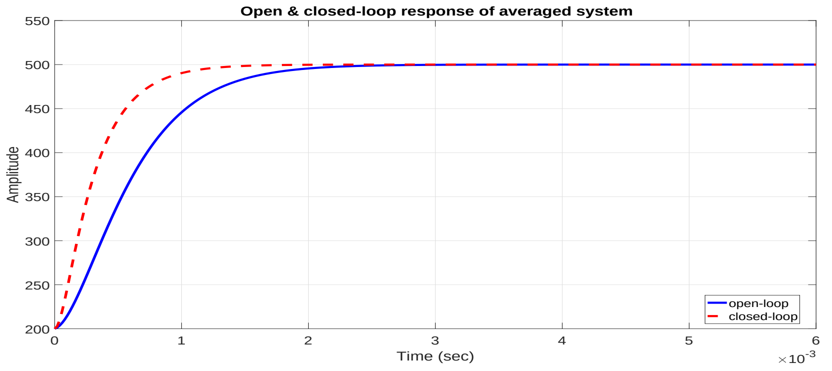

Figure 2.5: Open (blue solid line) and closed-loop (red dashed line) responses for the averaged system.

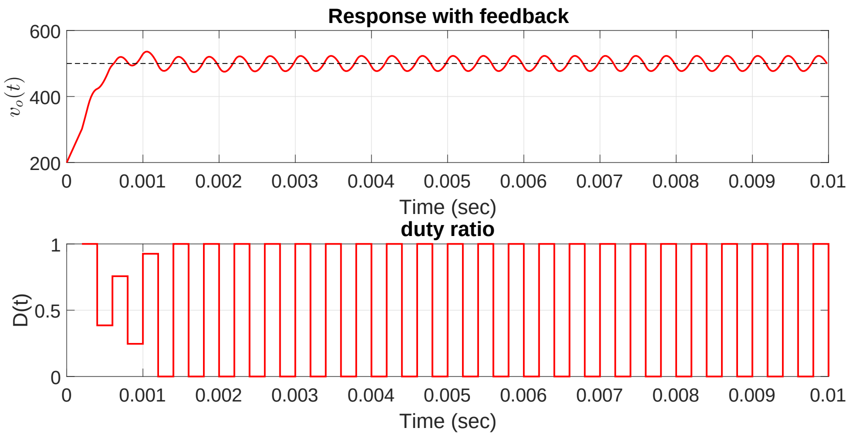

Figure 2.6: Short time simulation of the closed-loop system with the PD controller.

following nonlinear feedback function

$$D(t) = k_1(v_o^* - v_o(t))^3 + k_2 \frac{v_o'(t)}{\cosh(k_3(v_o^* - v_o(t)))} + k_c. \tag{2.11}$$

This function is not bounded between 0 and 1 in general, but we use a "bang-bang" strategy[5] to constrain it. The parameters k_i depend on the problem at hand. In general, a large deviation of v_o from v_o^* will 'activate' the first term in this expression,

[5]In general, the formula $D(t)$ can exceed the constraint $D(t) \in [0, 1]$. To prevent such a scenario, a "bang-bang" strategy is used. Basically, when $D(t) > 1$, we make it $D(t) = 1$ and when $D(t) < 0$, we make it $D(t) = 0$.

and k_1 can be used to shorten the response time. The second term in (2.11) should activate when v_o is near its reference value v_o^* with a large derivative $v_o'(t)$, this scenario will imply that $v_o(t)$ is going to overshoot. The parameter k_2 can be used to adjust the weight of this term, whereas k_4 determines the 'range of activation', i.e. the width of the $1/\cosh$ function. Finally, the third term k_c should be the duty ratio in steady state, e.g. $k_c = 0.5$ in our case.

PD control tuning method is still valid for k_1 and k_2, because they are still used to control the response time and the overshoot. However, since the transfer function is not defined for the full system, they can only be tuned based on the time-domain simulation. Hence, it requires more experiences and trial and errors.

Table 2.1 shows tuned parameter values for the standard PD controller and the nonlinear feedback function (2.11). Performance of the nonlinear feedback is depicted in Figure 2.7. Although the overshoot is around 20% of the reference output, the system is stabilized within 10 switching periods (2ms). In addition, the steady state error is within 1% of the reference output and the ripple frequency is 5kHz, which is exactly the switching frequency.

Table 2.1: Control parameter values for standard PD and nonlinear PD controllers.

	k_1	k_2	k_3	k_c
standard PD	0.0048	-1.3×10^{-6}	–	0.5
nonlinear PD	1.25×10^{-6}	2.5×10^{-4}	40	0.5

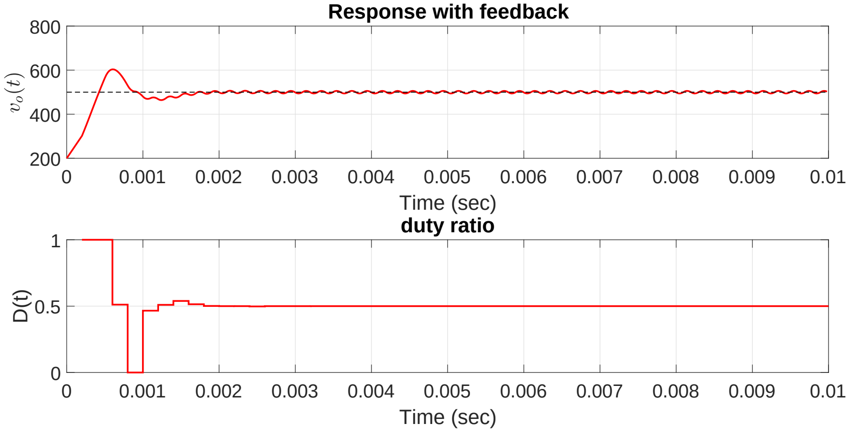

Figure 2.7: Performance of the nonlinear feedback controller.

Consider a different buck converter by decreasing the inductor from 1.3mH to 0.13mH. After retuning k_1, k_2 and k_4, the system can still be stabilized. The performance for this buck converter is plotted in Figure 2.8 with $k_1 = 10^{-9}, k_2 = 2.5 \times 10^{-8}, k_4 = 40$

and $k_c = 0.5$. The system stabilizes in 5 periods with an overshoot of around 700V and a steady state error of about 13% from the reference value. More careful tuning of the controller parameters could improve the performance.

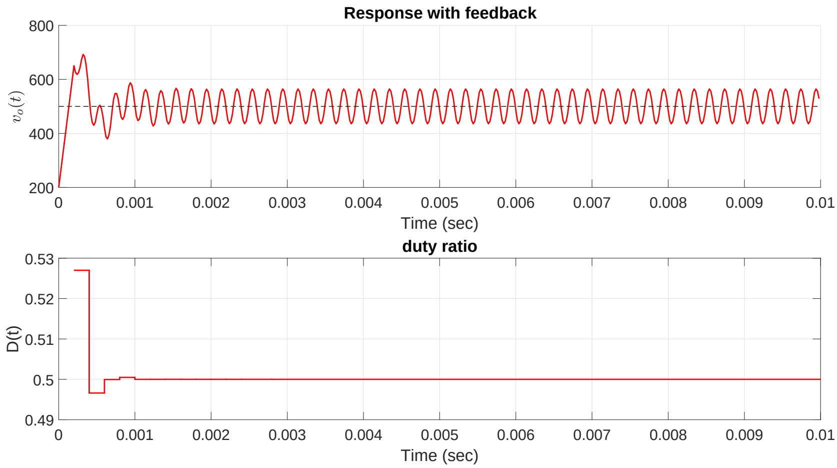

Figure 2.8: Performance of the nonlinear feedback controller for $L = 0.13$mH.

2.5 Conclusion and recommendations

High frequency DC-DC converters are widely used in power electronics for regulating the voltage from a source to a desired load. For efficient regulation, a DC-DC converter is designed using a power switch, an inductor, a capacitor, and a diode as the basic components. A feedback control system is used to maintain the output voltage constant at some reference voltage when the input voltage or the output current changes. This report discusses a technique to solve the ODE system associated to a buck-converter feedback control system.

An approach for analyzing the full ODE system is to consider the simplified averaged ODE system. In this work we provide criteria when this simplification provides an accurate description of the full system. Using the timescale $\tau = \sqrt{LC}$, we show that the averaged system remains consistent when $\tau >> T$, but becomes invalid if $\tau \sim T$.

Instead of looking at the averaged ODE system, we suggest solving the actual ODE system. A common approach for solving complicated ODE systems is to make use of numerical approximation methods. However, such techniques do not always guarantee accuracy, and often require a lot of computational time. Especially stiff ODEs are tough for numerical methods, since such systems often require very small time steps to achieve reasonable accuracy. In this report we have derived an analytical solution of the buck-converter feedback ODE system, which eliminates the need for approximation methods.

The key ingredient in a buck-converter feedback system is to determine duty ratios such that the output voltage converges to the reference output. Conventionally, a feedback function is defined as a linear function of output voltage. Unfortunately, the analysis of this linear feedback function is based on the averaged model and breaks down when applied to the full system. We have proposed a nonlinear feedback function, which appears to be working in cases where the classical feedback function breaks down.

A future suggestion is to use, whenever possible, analytical methods to solve similar ODE system. It not only helps in achieving accurate solutions and faster computational time, but also in the ease of implementation. Another possible extension is to use the analytical solution to find the optimal sequence of duty ratios. That is, using the principles and techniques of control theory, one can determine what sequence of duty ratios provides the fastest convergence towards the reference output voltage with the smallest overshoot.

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