Optimal dike heights around the IJsselmeer

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Abstract

In this note we show that the polytope associated with the IP model introduced by Zwaneveld and Verweij (6) is not integer. We also prove that, for a fixed number of dike segments, the problem can be solved in polynomial time. Similarly, we show that for a fixed number of allowed barrier heights, the problem can be solved in polynomial time.

1 Introduction

Protection against increasing sea levels is an important issue around the world. Optimal dike heights are of crucial importance to the Netherlands as almost 60% of its surface is under threat of flooding from sea, lakes, or rivers. This area is protected by more than 3500 kilometers of dunes and dikes. These dunes and dikes require substantial yearly investments of more than 1 billion euro (5).

Recently, Zwaneveld and Verweij (6) gave an integer programming model for a cost-benefit analysis to determine optimal dike heights that allows input-parameters for flood probabilities, damage costs and investment costs for dike heightening. The model by Zwaneveld and Verweij (6) is an improvement of the model proposed by Brekelmans et al. (1), who presented a dedicated approach with no optimality guarantee, and which was in turn an improvement of the original model by van Dantzig (4) from 1956. The latter was introduced after a devastating flood in the Netherlands in 1953.

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Our work is based on the IP model presented in a recent manuscript by Zwaneveld and Verweij (6), where the authors study the problem of economical optimal flood prevention in a situation in which multiple barriers dams and dikes protect the hinterland to both sea level rise as well as peak river discharges. Current optimal flood prevention methods (Kind (3), Brekelmans et al. (1)) only consider single dike ring areas with no interdependency between dikes. Zwaneveld and Verweij (6) present a model for a cost-benefit analysis to determine optimal dike heights with multiple interdependencies between dikes and barrier dams, and they also show that it can be solved quickly to proven optimality. The model was presented at the Study group Mathematics and Industry (SWI), taking place in Amsterdam in the last week of January 2017. It was our task at SWI 2017 to give a better understanding of the mathematical complexity of the model proposed by Zwaneveld and Verweij. The present report summarizes our approaches and results that were obtained during the week that SWI took place and the weeks after it.

We will follow the notation used in Zwaneveld and Verweij (6). Before going into the details of the problem, let us introduce some important terminology and the geographical configuration of the dikes in the Netherlands. A *dike segment* is a part of a dike that is protecting a region. It is possible that several segments protect the same area and in that case they are called a *dike ring*. In the Netherlands, dike ring areas and smaller dikes lie beneath the *Afsluitdijk*, sometimes denoted by the *barrier dam*, which is the most outer dike located in the north. The Afsluitdijk separates the North Sea and the IJsselmeer, an artificial lake.

In this paper we show that the polytope associated to the IP model introduced by Zwaneveld and Verweij (6) is not integer. Moreover, we present some sufficient conditions that allows the linear relaxation of the integer programming to avoid these non-integral points. We also prove that, for a fixed number of dike segments, the problem can be solved in polynomial time. Similarly, we show that for a fixed number of allowed barrier heights, the problem can be solved in polynomial time. This paper is organized as follows. In Section 2 we introduce the IP model that forms the subject of our investigations. In Section 3 we discuss integrality of the polytope. In Section 4 we propose an alternative approach to solve the problem by means of dynamic programming. Finally, in Section 5 we present a natural abstract version of the dike height problem, which allows for several variations and open problems.

2 IP Model formulation

In this section we present the model formulated in (6).

Throughout we use the following notation:

- D is the set of dike segments.
- H_D is the set of possible heights for a dike segment. For ease of notation, we do not let H_D depend on the dike segment, i.e., all dike segments have the same set of possible heights. We denote the height of a previous year by h_1, and that

of the current year by h_2. Likewise, H_B is the set of possible heights for the barrier dam and we denote the height of the barrier in the previous year by h_1^B, and that of the current year by h_2^B.

- T is the set of time periods at which changes to a dike segment can be made (e.g., one can assume that changes are scheduled per year), for simplicity we assume (with abuse of notation) $T = \{0, 1, \dots, T\}$.

The decision variables are:

- $CY(t, d, h_1, h_2) \in \{0, 1\}$. The variable being one meaning that dike ring d is updated in time period t from height h_1 up to height h_2. If $h_1 = h_2$ then this dike ring segment is not strengthened in period t and remains at its previous height. This decision variable is used for tracking investment (and maintenance) costs.

- $DY(t, d, h_2, h_2^B) \in \{0, 1\}$. It is one if at the end of period t the barrier dam has height h_2^b, and dike segment d is of height h_2. This variable is used to connect investments in dike segments (and the barrier dam) to expected damages. Another way to view it is that this variable linearizes the 0-1 variable $\left(\sum_{h_1} CY(t, d, h_1, h_2)\right)\left(\sum_{h_1^B} B(t, h_1^B, h_2^B)\right)$.

- $B(t, h_1^B, h_2^B) \in \{0, 1\}$. It is one if the barrier dam (i.e., the Afsluitdijk) is updated in time period t from height h_1^B up to h_2^B. If $h_1^B = h_2^B$ then the barrier dam is not strengthened in period t and remains at its previous height. This decision variable is used for bookkeeping investment (and maintenance) costs, flood probabilities and related expected damage costs of the barrier dam.

The input parameters are:

- $Dcost(t, d, h_1, h_2)$ = costs for investment and maintenance, if dike ring d is strengthened in time period t from h_1 to h_2. If $h_1 = h_2$, the dike ring segment is not strengthened and these costs only represent maintenance costs.

- $Dexpdam(t, d, h_2, h_2^B)$ = expected damage, i.e.,

$$Dexpdam(t, d, h_2, h_2^B) = prob(t, d, h_2, h_2^B) \times damage(t, d, h_2, h_2^B),$$

  where $prob(t, d, h_2, h_2^B)$ and $damage(t, d, h_2, h_2^B)$ are respectively the probability of failure and the expected damage cost (the latter given that there is a flooding) in period t given the height of the segment h_2 and the height of the barrier h_2^B. Note that it is assumed that both the probability of failure and the expected damage upon failure of dike segment d only depend on the height of segment d and that of the barrier dam.

- $Bcost(t, d, h_1^B, h_2^B)$ = costs for investment and maintenance, if the barrier dam is strengthened in time period t from h_1^B to h_2^B. If $h_1^B = h_2^B$, the barrier dam is not strengthened and these costs only represent maintenance costs.

- $Bexpdam(t, h_2^B)$ = expected damage of a flooding of the barrier dam, i.e. $prob(t, h_2^B) \times damage(t, h_2^B)$, here $prob(t, h_2^B)$ and $damage(t, h_2^B)$ are respectively the probability of failure and the expected damage cost (the latter given that there is a flooding), in period t given the height of the barrier h_2^B.

All input parameters are calculated in net present value of a certain year (i.e. 2015, which is the starting year for our calculations) and represent price levels in a certain year.

The IP model is:

minimize

$$\sum_{t\in T}\sum_{d\in D}\sum_{h_1\in H_D}\sum_{h_2\geq h_1} Dcost(t,d,h_1,h_2)\cdot CY(t,d,h_1,h_2)+ \tag{1}$$

$$\sum_{t\in T}\sum_{d\in D}\sum_{h_2\in H_D}\sum_{h_2^B} Dexpdam(t,d,h_2,h_2^B)\cdot DY(t,d,h_2,h_2^B)+ \tag{2}$$

$$\sum_{t\in T}\sum_{h_1^B\in H^B}\sum_{h_2^B\geq h_1^B}\Big(Bcost(t,h_1^B,h_2^B)+Bexpdam(t,h_2^B)\Big)\cdot B(t,h_1^B,h_2^B) \tag{3}$$

subject to

$$CY(0,d,0,0)=1, CY(0,d,h_1,h_2)=0 \quad \forall d\in D, h_1,h_2\in H_D, h_2\geq h_1\wedge h_2>0 \tag{4}$$

$$\sum_{h_1\leq h_2} CY(t-1,d,h_1,h_2)=\sum_{h_3\geq h_2} CY(t,d,h_2,h_3) \quad \forall t\in T_{>0}, d\in D, h_2\in H_D \tag{5}$$

$$\sum_{h_1\leq h_2} CY(t,d,h_1,h_2)=\sum_{h_2^B} DY(t,d,h_2,h_2^B) \quad \forall t\in T, d\in D, h_2\in H_D \tag{6}$$

$$B(0,0,0)=1, B(0,h_1^B,h_2^B)=0 \quad \forall h_1^B,h_2^B\in H_B, h_2^B\geq h_1^B\wedge h_2^B>0 \tag{7}$$

$$\sum_{h_1^B\leq h_2^B} B(t-1,h_1^B,h_2^B)=\sum_{h_3^B\geq h_2^B} B(t,h_2^B,h_3^B) \quad \forall t\in T\backslash\{0\}, d\in D, h_2^B\in H_B \tag{8}$$

$$\sum_{h_1^B\leq h_2^B} B(t,h_1^B,h_2^B)=\sum_{h_2} DY(t,d,h_2,h_2^B) \quad \forall t\in T, d\in D, h_2^B\in H_B \tag{9}$$

$$CY(t,d,h_1,h_2)\in\{0,1\} \quad \forall t\in T, d\in D, h_1\in H_D, h_2\geq h_1\in H_D \tag{10}$$

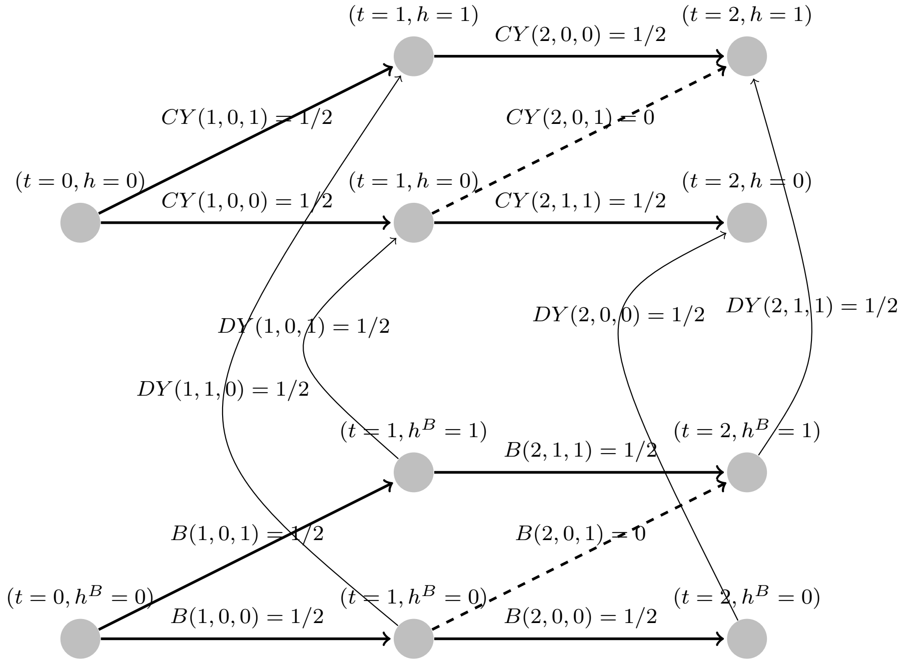
$$DY(t,d,h_2,h_2^B)\in\{0,1\} \quad \forall t\in T, d\in D, h_2\in H_D, h_2^B\in H_B \tag{11}$$

$$B(t,h_1^B,h_2^B)\in\{0,1\} \quad \forall t\in T, d\in D, h_2^B\geq h_1^B\in H_B \tag{12}$$

3 On the integrality of the polytope

In this section we show that, in general, there are vertices of the polytope defined by the linear relaxation of the constraints (when the integer values are considered to be in the interval $[0,1]$ instead of $\{0,1\}$), that have non integral coordinates.

Figure 1: Example of non-integer point.



The example involves the following sets indexing the variables.

- $T = \{0, 1, 2\}$
- one segment. Hence, we remove the dike index from all related variables.
- $H = \{0, 1\}$, $H_B = \{0, 1\}$

The point P, candidate to be a vertex of the polytope of the linear relaxation, has the following non-zero values:

- $CY(t, h_1, h_2)$: $CY(0,0,0) = 1$, $CY(1,0,1) = 1/2$, $CY(1,0,0) = 1/2$, $CY(2,1,1) = 1/2$, $CY(2,0,0) = 1/2$.
- $B(t, h_1, h_2)$: $B(0,0,0) = 1$, $B(1,0,1) = 1/2$, $B(1,0,0) = 1/2$, $B(2,1,1) = 1/2$, $B(2,0,0) = 1/2$.
- $DY(t, h_2, h_2^B)$: $DY(0,0,0) = 1$, $DY(1,0,1) = 1/2$, $DY(1,1,0) = 1/2$, $DY(2,1,1) = 1/2$, $DY(2,0,0) = 1/2$.

The example is summarized in Figure 3 where each arrow corresponds to one of the decision variables.

One can check that the example is a feasible solution (a point in the polytope). Indeed, the flow conditions are verified, as well as the equations linking the dummy variables DY and the CY's and B's (Equations (6) and (14)).

To argue that the point P is indeed a vertex of the polytope, we show that, for every line with non-zero direction vector $v = (x_0, \ldots, x_{14})$, and for every $\epsilon > 0$, either $P + \epsilon v$ or $P - \epsilon v$ is outside the polytope. Every coordinate x_i of v corresponds, uniquely, to a variable $B(\cdot)$, $CY(\cdot)$, or $DY(\cdot)$.

First observe that if x_i is the coordinate related to a variable that is either 0 or 1 in P, then $x_i = 0$, as otherwise, for any ϵ, either $P + \epsilon v$ or $P - \epsilon v$ would be outside of the polytope. Hence, the only x_i that may be non-zero, are those for which the coordinate i in P is in the open interval $(0, 1)$.

In our example, every equation involves at most 2 variables on each side of the equality, one of them being either 0 or 1. Hence the implications written below are forced by the previous observation. Assume, for instance, that the coefficient x_i corresponding to $B(2, 1, 1)$ in v is negative.

- Then, by the flow constraints (Equation (8)), the coefficient of $B(1, 0, 1)$ is negative.
- Then, by the flow constraints, the coefficient of $B(1, 0, 0)$ is positive.
- Then, by the flow constraints, the coefficient of $B(2, 0, 0)$ is positive.

Now, using the equations that link the variables B and DY, we obtain that the the coefficient of $DY(2, 1, 1)$ is positive, which implies that

- the coefficient of $CY(2, 1, 1)$ in v is positive.
- Then, by the flow constraints, the coefficient of $CY(1, 0, 1)$ is positive.
- Then, by the flow constraints, the coefficient of $CY(1, 0, 0)$ is negative.
- Then, by the flow constraints, the coefficient of $CY(2, 0, 0)$ is negative.

Observe now that this implies that the coefficient of $DY(2, 0, 0)$ has to be negative. However, let us now look at the coefficients of $DY(1, 0, 1)$ and the one corresponding to $DY(1, 1, 0)$.

If we use the links between the variables DY and B, the coefficients corresponding to the variables $DY(1, 0, 1)$ and $DY(1, 1, 0)$ in v have to be negative and positive respectively. However, if we look at the equations linking the variables DY and CY, the signs of the coefficients should have the opposite sign. Thus, these coefficients should be zero, implying that all the other coefficients have to be 0, which shows that no non-zero vector v exists.

The first coefficient involved in the argument was the one involving the variable $B(2, 1, 1)$. Since the implications described here involve all the non-zero variables of the point, and the implications are reversible, the result now follows.

3.1 Avoiding the non-integral points

We present here a sufficient condition on the objective function (1)–(3), that guarantees that either the linear relaxation of the integer program finds an integral point as a solution, or that there is an integral point in the optimal face and a procedure to find it.

Proposition 1. *Assume that, for every $h_2 \leq h'_2$ and $h_2^B \leq h_2'^B$ the objective function satisfies:*

$$D_{expdam}(t,i,h'_2,h_2^B) + D_{expdam}(t,i,h_2,h_2'^B) \geq D_{expdam}(t,i,h_2,h_2^B) + D_{expdam}(t,i,h'_2,h_2'^B) \tag{13}$$

and that, if $h_1 \leq h'_1$ and $h_2 \leq h'_2$, then, for every t,

$$B_{cost}(t,h_1,h'_2) + B_{cost}(t,h'_1,h_2) \geq B_{cost}(t,h_1,h_2) + B_{cost}(t,h'_1,h'_2) \tag{14}$$

and, for every t and d,

$$D_{cost}(t,d,h_1,h'_2) + D_{cost}(t,d,h'_1,h_2) \geq D_{cost}(t,d,h_1,h_2) + D_{cost}(t,d,h'_1,h'_2)\ . \tag{15}$$

Then, there is an optimal solution of the linear relaxation of the IP model in Section 2 with integer coordinates.

Proof of Proposition 1. The problem from Section 2 can be thought of as several intertwined min-cost flow problems (see Section 5), one for each dyke, and one for the barrier.

Let x_0 be a solution point given by the linear relaxation, and assume it is non-integral. Using the monotone relations (14) and (15), the paths of the non-zero flows that x_0 defines for each of the dykes and the barrier can be assumed to be completely ordered (as otherwise, the flow values on the edges might be modified while maintaining the value of the in flow and out flow at each vertex while not increasing the objective function). So, we obtain a layered flow, where no two flow-paths strictly cross between two layers of vertices corresponding to two different consecutive times. In particular, for each of the dykes d, we can talk about a top path U_d (the height profile being always larger or equal than all the other height profiles), and a bottom path L_d, whose heights are smaller or equal than all the other height profiles. There is also a top U_B and bottom L_B paths for the flow of the barrier.

Observe that, as x_0 is non-integral, at least one of the variables DY is non-integral (either not equal to zero or not equal to one). Let $DY_{\min}$ be the minimal distance of the non-integral variables to either 0 or 1.

Using (13) as a guideline repeatedly, we modify the variables DY from x_0 to create a new feasible solution x_1 in which the variables $DY(t,i,h_2,h_2^B)$ are "untangled". In particular, we can assume that

$$\begin{aligned} &DY_{x_1}(t,i,h_2(U_i),h_2^B(U_B)) = \\ &\qquad = \min\left\{\sum_{h_2} DY_{x_0}(t,i,h_2,h_2^B(U_B)), \sum_{h_2^B} DY_{x_0}(t,i,h_2(U_i),h_2^B)\right\} \end{aligned}$$

and that

$$DY_{x_1}(t,i,h_2(L_i),h_2^B(L_B)) =$$
$$= \min\left\{\sum_{h_2} DY_{x_0}(t,i,h_2,h_2^B(L_B)), \sum_{h_2^B} DY_{x_0}(t,i,h_2(L_i),h_2^B)\right\}$$

by reassigning some mass of the variables DY that are crossed. The remaining variables of x_0 are kept equal in x_1. The reassignment is done in a way to preserve the flow constraints, so x_1 remains feasible. By (13), x_1 has the same objective value as x_0, since x_0 is optimal.

Let $F_{\min}$ be the minimal difference to 0 or 1 of the flow through each L_d, U_d for every dyke d and L_B or U_B, which can be assumed to be the minimal value of

$$\min_{t,i}\left\{DY_{x_1}(t,i,h_2(U_i),h_2^B(U_B)), DY_{x_1}(t,i,h_2(L_i),h_2^B(L_B))\right\}$$

We note that x_1 is not a vertex of the polytope. Indeed, for any dyke d, we can pair up $L_d \leftrightarrow L_B$ and $U_d \leftrightarrow U_B$. Using (14) and (15), this pairing is well defined and consistent. In particular, we can redirect an ϵ flow ($0 < \epsilon \leq F_{\min}$) from each of the L_d to U_d and from L_B to U_B, or viceversa (the redirection of the flow should be done on each of the paths simultaneously, either from upper to lower paths, or from lower to upper ones). Since there exists a d (or B) for which the paths L_d and U_d differ, this flow-redirection by ϵ gives a different point on the polytope of feasible points and shows that x_1 is not a vertex of the polytope.

Furthermore, for every $\epsilon > 0$, the mentioned flow redirection should give the same value of the objective function (since otherwise x_0 would not have been an optimal solution). Hence we can choose to redirect the flow at our convenience; we redirect it so that the edge whose flow-value is $F_{\min}$ becomes either 0 or 1 (depending on whether its value is closer to 0 or to 1, if $F_{\min} = 1/2$, we arbitrarily redirect the flow either way). In particular, we have obtain a new solution x_2 where the number of edges with non-integral flow has been reduced, at least, by one. This procedure can be iterated until no non-integral flows are found. Therefore, an integral vertex of the polytope in the optimal face of the linear relaxation of the integer program is found. □

4 Alternative approaches

A feasible solution to the integer program presented in Section 2 can be interpreted as a choice of height $h^d(t)$ for each dike segment at each time period t, and a height $h^b(t)$ of the barrier dam. Abstractly, the cost of these height series can be written as a sum of cost terms which depend only on the 'upgrade' done in period t to segment d (i.e., a heightening of the dike, or merely the maintenance cost), we denote this by $\mathsf{cost}^d(h^d(t-1), h^d(t), t)$ for segment d, and by $\mathsf{cost}^b(h^b(t-1), h^b(t), t)$ for the barrier. Finally, there is also an expected damage cost for upgrading the dike and barrier to heights $h^d(t)$ and $h^b(t)$ in period t, denoted by $\mathsf{dam}^{d,b}(h^b(t), h^d(t), t)$. The problem modeled in Section 2 can thus be written in the following way:

$$\begin{aligned}&\mathsf{d-opt} =\\ \min\Big\{ &\sum_{t\in[T]} \mathsf{cost}^b(h^b(t-1), h^b(t), t) + \sum_{d\in D} \mathsf{cost}^d(h^d(t-1), h^d(t), t) + \mathsf{dam}^{d,b}(h^b(t), h^d(t), t)\\ &\text{s.t.}\quad h^d(t) \in H_D, h^b(t) \in H_B \text{ for } d \in D, t \in T\\ &\qquad h^d(t) \geq h^d(t-1) \text{ for } d \in D, t \in T\\ &\qquad h^b(t) \geq h^b(t-1) \text{ for } t \in T\Big\}\end{aligned}$$

The linear relaxation of the integer programming model presented in Section 2 can be solved in time polynomial in $|D|, |T|, |H_D|$, and $|H_B|$. However, in general there is no guarantee that the returned solution is integral, see Section 3. In the next two sections we describe two different approaches to solving this problem. Both approaches have the benefit of solving the integer problem exactly. However, this comes at a cost: both approaches give a polynomial time algorithm only if one of the parameters is regarded as a constant. The first approach is to solve the integer program by ways of a dynamic program. The second approach comes down to enumerating all possible height profiles of the barrier dam, and for each profile solving shortest path problems on small graphs.

4.1 Dynamic programming

There are two key observations to be made. First, the second part of the objective function decomposes naturally into a sum of $|D|$ terms, each of which depends only on the barrier height and one segment. Secondly, for each time period the cost only depends on the dike/barrier heights at times $t-1$ and t. Together this allows us to solve the problem using a dynamic program. The recursion will be on the time period. We maintain the following table: $\mathsf{opt}(h^b, \mathbf{h^s}, t)$ for all $t \in T, h^b \in H_B, \mathbf{h^d} \in (H_D)^D$. The interpretation is as follows, $\mathsf{opt}(h^b, \mathbf{h^d}, t)$ is equal to the minimum cost made, up to time t, if the barrier and segments are of height h^b and $\mathbf{h^d}$ at time period t respectively. We can compute the entries of this table as follows:

$$\begin{aligned}\mathsf{opt}(h^b, \mathbf{h^d}, t) = \min\Big\{&\mathsf{opt}(h^b - i^b, \mathbf{h^d} - \mathbf{i^d}, t-1) + \mathsf{cost}^b(h^b - i^b, h^b, t) +\\ &\mathsf{cost}(\mathbf{h^d} - \mathbf{i^d}, \mathbf{h^d}, t) + \mathsf{dam}(h^b, \mathbf{h^d}, t) :\\ &h^b - i^b \in H_B, \mathbf{h^d} - \mathbf{i^d} \in (H_D)^{|D|}\Big\}\end{aligned}$$

It follows that each entry of the table can be computed in time $\mathcal{O}(|H_B||H_D|^{|D|})$. Hence, all entries of the table can be filled in time $\mathcal{O}\big((|H_B||H_D|^{|D|})^2 \cdot |T|\big)$. Using the interpretation of $\mathsf{opt}(h^b, \mathbf{h^d}, t)$ it follows that

$$\mathsf{d-opt} = \min_{h^b \in H_B, \mathbf{h^d} \in (H_D)^{|D|}} \mathsf{opt}(h^b, \mathbf{h^d}, T)$$

This shows the following result:

Theorem 4.1. *One can determine* $\mathsf{d}-\mathsf{opt}$ *in time* $\mathcal{O}\big((|H_B||H_D|^{|D|})^2 \cdot |T|\big)$.

4.2 Shortest paths

In the previous section we have seen an algorithm for computing the optimal dike/barrier height profiles which has polynomial runtime for a fixed number of dike segments, in this section we present a different algorithm, based on shortest paths, that runs in polynomial time when the number of possible barrier heights is fixed. We present an algorithm that computes $\mathsf{d}-\mathsf{opt}$ in time

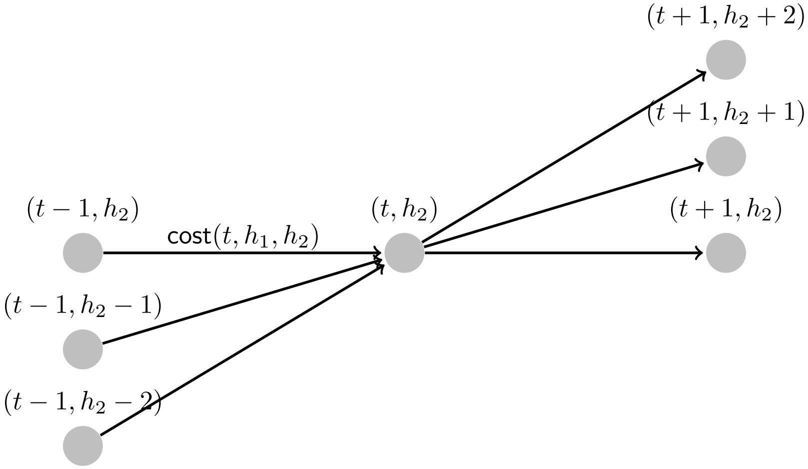
$$\mathcal{O}\left(\overbrace{|D|}^{\#\text{ segments}} \cdot \underbrace{(T\cdot|H_D|)^2}_{\text{Complexity shortest path}} \cdot \overbrace{T^{|H_B|}}^{\#\text{ barrier height profiles}}\right).$$

To illustrate the basic idea we first discuss the algorithm for the setting of one dike segment and no barrier, we then add a barrier dam and from that the generalization to multiple dike segments and barriers easily follows.

4.2.1 One dike segment, no barrier

First consider the situation with only one dike segment and no barrier. In this case the problem of minimizing the cost at time period T becomes equivalent to finding a shortest p-q path in the following graph. The source $p = (0, 0)$ is the initial height of the dike at time 0. Then, for each time $t \in [T]$ and each possible height of the dike h, we define a node (t, h). Finally we define a sink node q. The edges are defined as follows. We first add an edge between $(0, 0)$ and $(1, h)$ for each $h \in H_D$, with weight $\mathsf{cost}(0, h, 1)$, similarly for each $t \in [T]$ and height pair $h_1 \leq h_2$ there is an edge from $(t-1, h_1)$ to (t, h_2) with weight $\mathsf{cost}(h_1, h_2, t)$ equal to the financial cost associated to the decision of raising the dike segment from height h_1 to h_2 in time period t. Notice that since there is no barrier, we can assume that the expected damage cost $\mathsf{dam}(t, h)$ are incorporated in $\mathsf{cost}(h_1, h_2, t)$. Finally, the nodes (T, h) are all connected to the sink q. In the figure below the incoming and outgoing arcs of a node (t, h_2) are sketched for some $0 < t < T$ and $h_2 \in H_D$. One observes that, indeed, the shortest p-q path corresponds to the best strategy of heightening this dike segment.

Recall, the shortest p-q path in a graph $G = (V, E)$ with nonnegative edge weights can be found in time $\mathcal{O}(|V|^2)$ using Dijkstra's algorithm.



4.2.2 One dike segment, a barrier

We now consider the case of a single dike segment and a barrier. The observation we need to make is that the total financial cost incurred by upgrading the dike segment from height h_1 to height h_2 in time period t no longer only depend on the dike segment, they also depend on the height of the barrier at time point t. This means that we cannot solve a shortest path problem for the barrier and dike segment separately: the costs on the dike segment graph depend on the path chosen in the barrier graph.

The key idea is that if we fix the height of the barrier at each time t, then we reduce to the previous setting where all the costs are known. Hence, the optimization problem $\mathsf{d}-\mathsf{opt}$ can be solved by minimizing over the possible height profiles $h^b(t)$ of the barrier over time, the minimum cost of a p-q path in the network defined in the previous section (using the costs associated to $h^b(t)$) plus the cost of implementing height profile $h^b(t)$. The outer minimization over the possible height profiles $h^b(t)$ is performed by enumeration, which takes time roughly $T^{|H_B|}$. This means that the optimal investment strategy for both the dike segment and barrier can be found in time

$$\mathcal{O}\left((T \cdot |H_D|)^2 \cdot \binom{T}{|H_B|}\right) = \mathcal{O}\left((T \cdot |H_D|)^2 \cdot T^{|H_B|}\right).$$

4.2.3 Multiple dike segments and a barrier

The approach of the previous section easily generalizes to the setting of multiple dike segments and a barrier. Once a height profile $h^b(t)$ of the barrier dike is fixed, the optimal height profiles of each of the different dike segments can be computed independently. Hence the problem of finding the optimal investment strategy for multiple dike segments and a barrier can be solved in time

$$\mathcal{O}\left(|D| \cdot (T \cdot |H_D|)^2 \cdot T^{|H_B|}\right).$$

This approach generalizes to the setting of multiple barriers and dike segments (where the costs of a dike segment at time t may depend on the height of several barriers). The complexity will be of the form

$$\mathcal{O}\left(|D| \cdot (T \cdot |H_D|)^2 \cdot T^{|H_B||B|}\right),$$

where $|B|$ is the number of barriers. One should note that the above approach assumes the same discretization in time of the barrier and dike segments. It seems reasonable to assume a coarser discretization for the barrier of say T_B steps, this would reduce the above-mentioned formula to

$$\mathcal{O}\left(|D| \cdot (T \cdot |H_D|)^2 \cdot (T_B)^{|H_B||B|}\right).$$

5 An abstraction of the problem

In this section we present a natural abstract version of the dike height problem, which allows for several variations and questions, which we believe have not been considered in the literature before. We believe that studying these variations may shed more light on the complexity of the dike height problem.

In the dike height problem we essentially have two directed graphs where each path in one of the two graphs (the one modeling the height of the barrier dam) influences the cost of arcs in the other graph. It is not difficult to show that if we were to allow any kind of influence of the path in the one graph on the cost of arcs in the other graph, the problem would automatically become NP-hard. Indeed, one can easily show that in this case the problem contains the problem of finding two vertex disjoint paths in a directed graph, which is NP-complete (2).

For this reason, we consider the following restricted problem.

Definition 5.1. For $k \in \mathbb{N}$, a *k-layered* graph is a directed graph $D = (V, A)$ such that V is partitioned into *layers* $V = V_0 \cup V_1 \cup \ldots \cup V_k \cup V_{k+1}$ such that each $a \in A$ is from V_i to V_{i+1} for some $i = 0, \ldots, k$ and where V_0 and V_{k+1} both consist of a single vertex and where $|V_1| = |V_2| = \cdots = |V_k|$. We denote the arcs between V_i and V_{i+1} by $A[V_i, V_{i+1}]$ and we refer to $|V_1|$ as the *partition size.*

Definition 5.2 (Minimum intertwined-cost path).
Input: two k-layered graphs $G_1 = (V_1, A_1), G_2 = (V_2, A_2)$, with partitions $V_i = V_1^{(i)} \cup \ldots \cup V_k^{(i)}$, respectively, cost functions $c_1 : A_1 \to \mathbb{R}_{\geq 0}$, $c_2 : A_2 \to \mathbb{R}_{\geq 0}$ and for each $i = 1, \ldots, k$ a map $m_i : V_i^{(2)} \times A[V_{i-1}, V_i] \to \mathbb{R}_{\geq 0}$.

Given a path $P_2 = (a_1, v_1, a_2, v_2, \ldots, a_k, v_k, a_{k+1})$ from $V_0^{(2)}$ to $V_{k+1}^{(2)}$ and a path $P_1 = (a'_1, \ldots, a'_{k+1})$ from $V_0^{(1)}$ to $V_{k+1}^{(1)}$, we define the *cost* of the pair (P_1, P_2) as

$$\text{cost}(P_1, P_2) = \sum_{i=1}^{k+1}(c_1(a_i) + c_2(a'_i)) + \sum_{i=1}^{k+1} m_i(v_i, a_i).$$

Output: the minimum cost of a pair of paths (P_1, P_2) over all pairs and a pair of paths (P_1^*, P_2^*) attaining this minimum.

In the Minimum intertwined-cost problem, the dependence of $\text{cost}(P_1, P_2)$ on the path P_2 is linear in the edges of P_2. It is not difficult to see that the dike height problem in Section 4.2.2 can be modeled as a special case of the Minimum intertwined-cost path problem, where for both graphs the arcs between V_i and V_{i+1} are somewhat restricted. More precisely, if we identify each $V_i^{(2)}$ $(i = 1, \dots, k)$ with $H_B =: \{h_1, \dots, h_t\}$ then the only arcs that are present are of the from (h_i, h_j) with $h_i \leq h_j$. This particular fact allowed us in Section 4.2.2 to give an algorithm for the problem, which runs in polynomial time if we consider the size of the sets in the partition of the vertices of the second graph as a constant. Clearly if the bipartite graphs between $V_i^{(2)}$ and $V_{i+1}^{(2)}$ are complete, then this dynamic programming approach will not work. It would be interesting to find out if some other approach may yield an efficient algorithm.

We end this section with some concrete questions.

Question 1. *Is the Minimum intertwined-cost path problem NP-hard?*

If this question has a positive answer, then it makes sense to consider the following questions.

Question 2. *Under which conditions on the bipartite graphs $G_j[V_i^{(j)}, V_{i+1}^{(j)}]$, $(j = 1, 2,$ $i = 1, \dots, k)$ is there a polynomial time algorithm for the Minimum intertwined-cost path problem?*

Question 3. *Suppose the partition size of G_2 is constant. Under which conditions on the bipartite graphs $G_j[V_i^{(j)}, V_{i+1}^{(j)}]$ $(j = 1, 2,$ $i = 0, \dots, k)$ is there a polynomial time algorithm for the Minimum intertwined-cost path problem?*

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