Energy Consumption of Trains

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Abstract

In this report, we consider a problem on energy minimisation of trains proposed by Nederlandse Spoorwegen (NS). Our results include a quick heuristic to compute the energy consumption for a given time table as well as a heuristic to find a timetable which is more energy efficient.
KEYWORDS: Energy minimisation, Timetabling, Heuristic algorithm

1 Introduction

We consider the problem proposed by Nederlandse Spoorwegen (NS) at the Study Group Mathematics with Industry 2016, held at Radboud University, Nijmegen. NS is a Dutch passenger railway operator and provides domestic and international rail services, which makes the company one of the largest consumers of electricity in the Netherlands. Due to environmental considerations and the quality of service for the passengers, NS seeks methods to reduce carbon dioxide CO_2 emissions and to improve the efficiency of the railway system.

Figure 1 shows that the energy optimal way of going from one station to the next (when there are no intermediate constraints). The behaviour of a train is described by four driving regimes: *accelerating*, *cruising* (maintaining constant speed), *coasting* (driving without using energy), *braking*. This is derived using Pontrayagin's Maxium Principle Pontryagin et al. (1962) cf. Howlett (1996); Khmelnitsky (2000); Liu and Golovitcher (2003); Scheepmaker and Goverde (2015a). To find the energy optimal profile one then needs to determine the points x_1, x_2 and x_3 depending on how much time is scheduled to go from one station to the next.

In this project, the main objective is to obtain understanding of how modifications in timetabling can even out the electricity demands, and hereby increase energy efficiency. In fact, this consists of (at least) two subproblems.

- Problem 1: Given a timetable, find the the most energy efficient way for the trains to drive from station to station.

- Problem 2: Find a timetable that uses least energy.

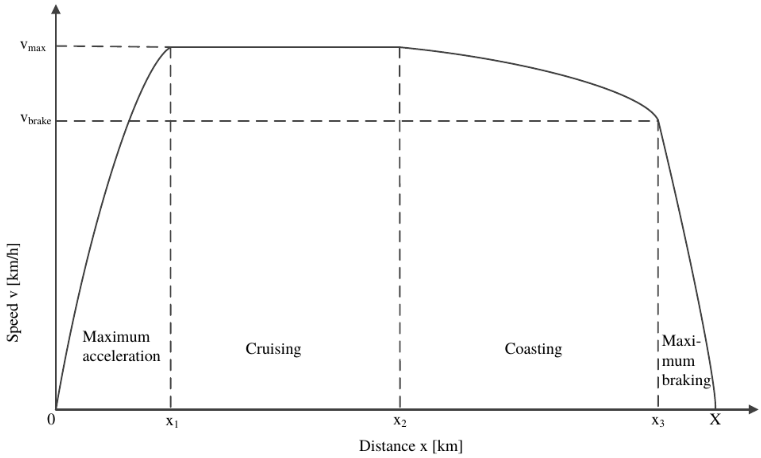

Figure 1: Optimal velocity profile of a basic energy-efficient driving strategy on a level track with switching points between driving regimes at x_1, x_2 and x_3. Courtesy of Gerben M. Scheepmaker(Scheepmaker, 2013).

In view of Figure 1, it looks that one just has to determine the x_i to solve Problem 1 for a given timetable. However, for a journey between two stations there are a lot of additional constraints that are not visible in the public timetable. For example, there are constraints saying that two trains may not pass the same point within 3 minutes.

The paper is organized as follows: In Section 2, we formulate these problems concretely. In the remainder of the paper we focus on our attempts to find a solution to these problems. In Section 3, we look at a heuristic solution for computing the optimal energy profile; i.e., a solution to Problem 1. This heuristic is also tested on a realistic data set from NS. In Section 4, we take a numerical approach to compute the optimal energy profile for a realistic data set from NS and use this to find an improved timetable. We close with discussion in Section 5.

2 Formulation of the problem

In this section, we consider a basic energy-efficient train control model which is the problem of driving along a flat track within a given time T. The train speed $v(t)$ at time t is governed by an energy functional $F(t)$ and a resistance force $r(v)$ according

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to the Newton force equilibrium

$$\rho m v' = F(t) - r(v(t)), \tag{1}$$

where $v' = \frac{\mathrm{d}v}{\mathrm{d}t}$ is the derivative of velocity to time, m is the train mass, ρ the dimensionless rotating mass factor (Brünger and Dahlhaus, 2007). The resistance force $R(v)$ is given by the Davis equation

$$r(v) = r_0 + r_1 v + r_2 v^2. \tag{2}$$

Here r_1, r_2 and r_3 are non-negative coefficients (Davis, 1926). The energy consumption to be optimised is given by

$$E = \int_0^T F^+(t)v(t)dt, \tag{3}$$

where F^+ denotes the nonnegative part of F. That is we do not assume that the train can gain energy from braking, contrary to e.g. Scheepmaker and Goverde (2015b).

As mentioned in the introduction, if there are no further constraints between station A and B, then Figure 1 gives the energy optimal speed profile. However, generally there are additional constraints to be met between station A and B. A journey consists of *events*. An event should be thought of as 'train α passes junction x' or 'train β arrives at station y' etc. For each event i, there is a variable t_i saying at what time in minutes this event takes place. There is one catch however. Since the timetable should be periodic, these times are to be prescribed modulo 60 minutes. Then there are constraints prescribing how certain events relate to each other; they are all of the form

$$l_{i,j} \leq (t_i - t_j) \mod 60 \leq u_{i,j}, \tag{4}$$

saying that event j should take place at least $l_{i,j}$ minutes later than event i and not later than $u_{i,j}$ minutes after event i. For example, this could encode that the time that train β passes junction x should be at least 3 minutes later than the time that train α passes junction x. When designing a timetable it is exactly the modularity of the constraints that makes this a really difficult task. So one usually modifies a feasible solution to obtain a better solution. In particular, fixing a feasible solution, i.e. a timetable that satisfies the constraints, one can get rid of the modularity constraints and then the constraints (4) all of a sudden look much nicer: they are totally unimodular; see Schrijver (1998) for details on totally unimodularity and its use in optimisation.

3 Heuristic solution

In the case of a single segment the optimal solution consists of four different phases: acceleration, cruising, coasting and braking, in this order (Howlett and Pudney, 1995). The optimal length of each phase can be found by a simple line search (e.g., using the cruising speed as parameter).

The optimal solution in the case of multiple segments along a railway track is fundamentally different and more difficult to obtain, especially if one is interested in a computationally efficient solution. In the following we will suggest an approximate solution based on heuristic reasoning. The motivation for this is the following theorem, which for a lack of better name we call the *friction theorem*.

3.1 The friction theorem

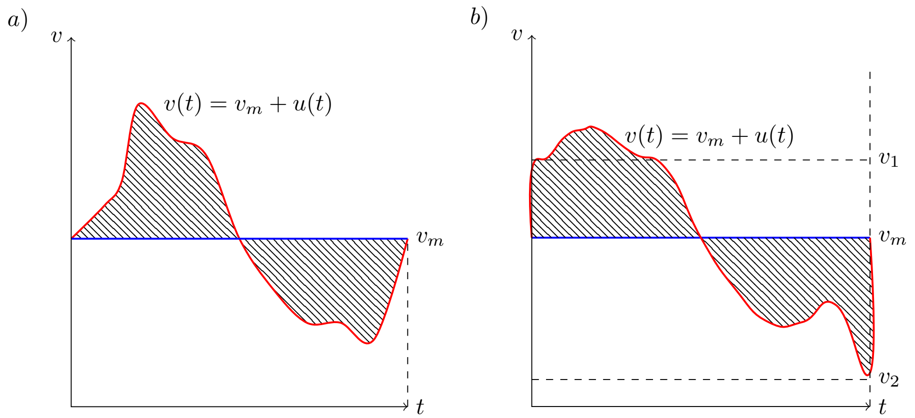

Figure 2: Illustration of the friction theorem. a) In velocity space the area under a curve corresponds to the distance travelled. The constant trajectory with the mean velocity v_m (blue curve) needs less energy than any other equal-area trajectory $v(t) = v_m + u(t)$ (covering the same distance) starting and ending with v_m (red curve). b) A consequence of the friction theorem is a bound on the maximum energy that can be saved by an equal-area trajectory starting at velocity $v_1 > v_m$ and ending at $v_2 < v_m$.

Theorem 3.1. *The optimal way of getting across a distance x_1 in time t_1, when nonlinear friction is acting and when starting and ending with the average speed $v_m = x_1/t_1$, is by traveling all the way at the average speed.*

The proof is based on the intuition that nonlinear friction forces do not average out across the trajectory. We will show this for the Davis model of friction (2) that is relevant for railway problems.

Proof. We decompose the work dW performed on the system by external forces into a contribution dR due to the frictional resistance and a contribution dT used to raise or lower the kinetic energy: $dW = dR + dT$. The kinetic energy is the same at the beginning and at the end of the trajectory, therefore $\int dT = \Delta T = 0$. The total work done on the system is therefore equal to the work done against friction, and amounts to

$$W = \int \mathrm{d}W = \int \mathrm{d}R = \int_0^{x_1} r(v)\,\mathrm{d}x = \int_0^{t_1} r(v)v\,\mathrm{d}t. \tag{5}$$

Decompose the trajectory in velocity space into $v(t) = v_m + u(t)$, where $v_m = x_1/t_1$ is the mean velocity (Figure 2a). Compared with the mean trajectory $v(t) = v_m$, the difference in energy expended is

$$\begin{aligned} \Delta W &= \int_0^{t_1} r(v_m + u)(v_m + u)\,\mathrm{d}t - \int_0^{t_1} r(v_m)v_m\,\mathrm{d}t, \qquad (6) \\ &= r(v_m)\int_0^{t_1} u\,\mathrm{d}t + \int_0^{t_1} (r_1 u + r_2 u^2 + 2r_2 u v_m)(v_m + u)\,\mathrm{d}t. \end{aligned}$$

The first term is zero due to the constraint on the distance travelled (which is equal to the area under the velocity trajectory),

$$\int_0^{t_1} (v_m + u)\,\mathrm{d}t = x_1 = \int_0^{t_1} v_m\,\mathrm{d}t \quad \Rightarrow \quad \int_0^{t_1} u\,\mathrm{d}t = 0. \tag{7}$$

The remaining term amounts to

$$\begin{aligned} \Delta W &= \int_0^{t_1} (r_1 u + r_2 u^2 + 2r_2 u v_m)(v_m + u)\,\mathrm{d}t, \qquad (8) \\ &= (r_1 + 3r_2 v_m)\int_0^{t_1} u^2\,\mathrm{d}t + r_2\int_0^{t_1} u^3\,\mathrm{d}t, \end{aligned}$$

where we have used Eq. 7 again to simplify. Writing the remainder as

$$\Delta W = r_1 \int_0^{t_1} u^2\,\mathrm{d}t + r_2 \int_0^{t_1} (u + 3v_m)u^2\,\mathrm{d}t, \tag{9}$$

and using that $|u| \le v_m$, shows that $\Delta W \ge 0$. □

3.2 Consequences of the friction theorem

Theorem 3.1 has important consequences, in combination with the constraint on distance travelled. Consider first the journey along a single segment or track, that starts at a velocity below the mean velocity v_m and is supposed to finish at a velocity similarly below v_m. Because of the constraint on distance travelled, there needs to be some acceleration in between and the trajectory follows the well-known optimal shape with up to four phases (acceleration, cruising, coasting, braking) in succession.

A trajectory that includes coasting needs to accelerate longer and the final velocity at the end of the segment will be lower than when only cruising (Figure 3). If the loss in kinetic energy due to coasting leads to the coasting ending on the braking curve, some energy has been saved. However, if the loss in kinetic energy due to

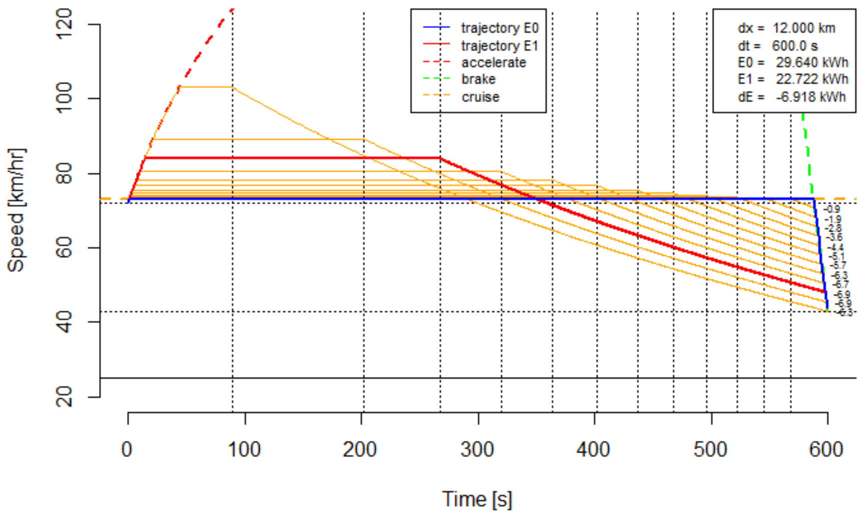

Figure 3: Increasing the energy efficiency by coasting, when there is braking at the end. The blue curve corresponds to travel without coasting. The red and orange curves show alternative trajectories that use coasting to reduce the energy expenditure. Longer coasting needs higher initial acceleration and results in lower velocities. The energy saved with respect to the blue curve is given on the right (in kWh) for each of these curves. In this example about 15 percent of the energy can be saved.

coasting needs to be compensated, i.e., if an additional acceleration is (during this or a following segment) needed because of the coasting, then the friction theorem tells us that this is energetically unfavourable. It is better then to reduce the amount of coasting (by reducing the cruising speed and increasing the cruising phase) until the loss in velocity has no consequences. In other words: coasting can be used to reduce the energy expenditure only when it *replaces braking*, not when it incurs additional acceleration later on. Note that in practice, the potential gain of this is eventually limited by the increasingly unfavourably loss due to the nonlinear behaviour of the friction, cf. Figure 4.

This is the main difference with the situation where only a single segment needs to traversed. In that case, coasting could potentially reduce the energy to zero, if this would result in exactly the right distance travelled. In the case of multiple segments, however, coasting should only reduce the kinetic energy if the train is travelling too fast for the next segment anyway, such that braking would be needed otherwise.

The friction theorem also gives us a bound on the maximum energy saving:

Corollary 3.2. *The energy that can be saved by coasting during a trajectory starting at $v_1 \geq v_m$ and ending at $v_2 \leq v_m$ is at most equal to the kinetic energy difference because of the difference in starting and ending velocities,*

$$\Delta E \leq \frac{1}{2} m \left(v_1^2 - v_2^2 \right) . \tag{10}$$

Proof. The friction theorem shows that $\Delta E \leq 0$ for a modified trajectory that includes a (hypothetical, instantaneous) initial and final acceleration from v_m to v_1 and from v_2 to v_m, respectively (Figure 2b). Subtracting the difference in kinetic energy results in Eq. 10 for the trajectory starting at v_1 and ending at v_2. □

What is the optimal amount of coasting? There is no simple, definite answer to this, as it depends on the interplay between the nonlinearities in the friction $r(v)$ and the geometric properties of the trajectory. The optimal trajectory balances replacing as much cruising (work against frictional losses) as possible with coasting (no work) with the increased work during the initial acceleration and (shorter) cruising phase. In practice it seems often to be the case that close to the least amount of cruising leads to the best energy balance (Figure 3).

This leads to the following heuristic, where the phases in brackets can be missing:

- Where possible, replace *cruising + braking* with *accelerating + (cruising) + coasting + (braking)*. If the best curve to follow cannot be determined (e.g., because of the need for a highly efficient method that cannot optimise the cruising speed), use the highest cruising velocity ending on the braking curve.

What happens if the train travels too fast initially? It is always possible to satisfy the constraint on distance by first braking, then cruising, followed by accelerating or braking, as necessary. Similar to the the first case, it is possible to relax this solution by the following heuristic, thereby also improving energy efficiency (Figure 4):

- Replace *braking + cruising* with *(braking) + (cruising) + coasting + accelerating*. If the best curve cannot be determined, use the one with the highest cruising speed (and thereby the lowest speed immediately after coasting).

The energy saving in this case is typically much lower than for the first case and only significant when the acceleration at the end of the segment is very large.

3.3 A reference solution for multiple segments

Solving for the optimal cruising velocities in the above cases of a single segment (Figure 3-4) is not difficult. A straightforward algorithm uses a double loop where the outer loop optimises the cruising speed for the best saving in energy and the inner loop searches for the corresponding length of the cruising phase, in order to fulfill the constraint on distance travelled. As both loops search for a minimum in one dimension, Brent's algorithm or a variant thereof can be used (Press et al., 2007).

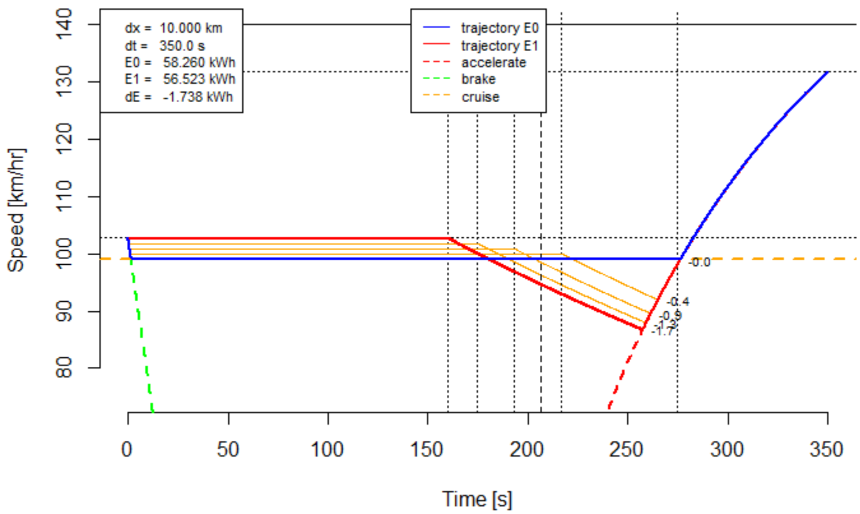

Figure 4: Increasing the energy efficiency by coasting. This is the case with acceleration at the end. This results in the need for an initial drop (braking) for the reference curve (blue) without coasting. The red and orange curves show alternative trajectories that use coasting to reduce the energy expenditure. Longer coasting needs higher initial velocity and results in lower velocities. The energy saved with respect to the blue curve is given on the right (in kWh) for each of these curves. In this example about 3 percent of the energy can be saved.

The main question is how to optimise the energy across multiple segments with intermediary constraints on times and distances. The above suggests a simple, heuristic solution:

1. The first segment is treated in a special way. The train accelerates to the velocity needed to cross the rest of the segment just by cruising. The rest of the segment is then treated as a new segment according to the following procedure.

2. Each segment starts with the mean velocity needed to cross it only by cruising, which would be optimal if not for the differences in mean velocity between segments.

3. Each segment anticipates the subsequent segment and at its end either accelerates or brakes the train to the mean velocity of the following segment.

4. If this cannot be achieved (due to time/distance constraints), then the train accelerates or brakes as much as possible, and the next segment is split into two phases. In the first part the train continues to accelerate or brake until the mean velocity for the remaining second part is reached. (The point where this happens needs to be calculated in an iterative way, since shortening the second part changes its mean velocity). The second part is then treated as a new segment.

5. If braking is needed at the end of the current segment, this means that an additional acceleration is needed at the beginning. Coasting is additionally introduced to relax this situation to a more energetically favourable one, reducing the amount of braking (as in Figure 3).

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3.4 Example solution

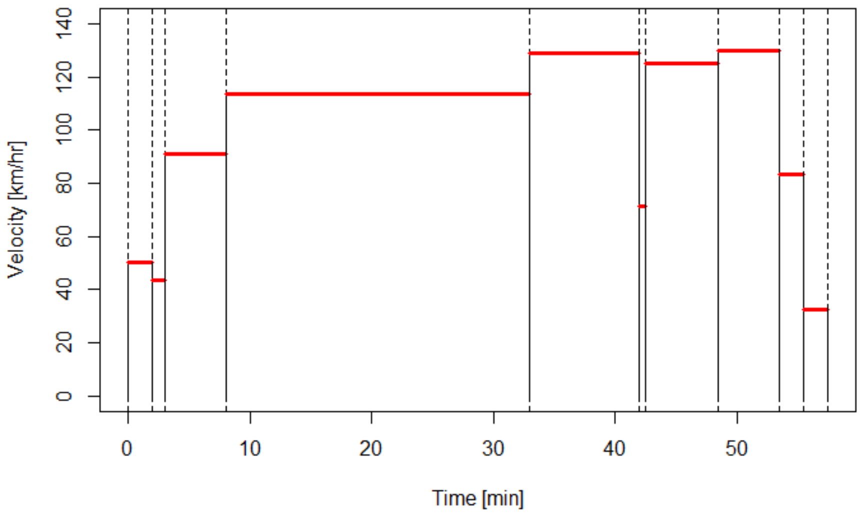

Figure 5: Example track. The mean velocity for each segment is shown. Large differences in these velocities potentially lead to energy-inefficient journeys.

The track between Groningen and Zwolle was used for this example, consisting of in total 10 segments. The mean speed along the segments of the track varies considerably (Figure 5). The train data was compiled from data given by (Scheepmaker, 2013) and NS. The timetable entries were rounded to the minute and are therefore not completely realistic. In fact, the timetable had to be slightly adjusted in order to be feasible.

A reference solution with only cruising needs 582.6 kWh for this track. Solving for the solution with the above algorithm leads to an energy consumption of 550.1 kWh, which is an improvement of 5.6 percent. This value is not the true minimum, but it seems unlikely that the energy expenditure could be further reduced by very much. Most improvements were obtained during the longest segments, where coasting could be used for a significant part of the journey (Figure 6, panels 5 and 8).

3.5 Discussion

This section shows one way of quickly constructing an approximate solution to the most energy-efficient journey along a railroad track with multiple segments (checkpoints). The method is sufficiently fast that it can be used to evaluate thousands of tracks, i.e., a complete timetable, in a reasonable time.

The computations for this section have been made with a simple, straightforward implementation in the system for computational statistics R (R Core Team, 2015). Solving for a single track and plotting the solution takes a few seconds only. Implementing the method in a compiled language and optimizing the code should result in runtimes of a few microseconds per track, which is suitable for applications such as timetable optimisation.

The timetable constrains the solution very much. Especially the occurrence of large differences in mean velocities for different segments of a journey lead to inefficient voyages, due to the need for braking and re-acceleration. Coasting can reduce some of these losses, but often only partially. It seems likely that more energy can be saved by adjusting the timetable (if possible) then by further optimizing the individual journeys for the given timetable beyond what has been shown here. As a next step one should therefore investigate how changes in the timetable affect the energy expenditure.

4 Towards better timetable

4.1 Optimal Energy for a given timetable

Two stops, A and B, are positioned at distance X apart from each other. We consider a train going from A to B in time T. The velocity at A,B is zero, $v(t) = 0,\ t = t_A, t_B,\ T = t_B - t_A$. In the current setup of the problem the timetable is fixed. That is to say a train has to pass prescribed intermediate points at distances x_i from A at specific times t_i, $i = 1, \ldots, N$. Without loss of generality we may consider both the journey time and the distance to be unities: $X = 1, T = 1$, so that $t_A = 0, t_B = 1$

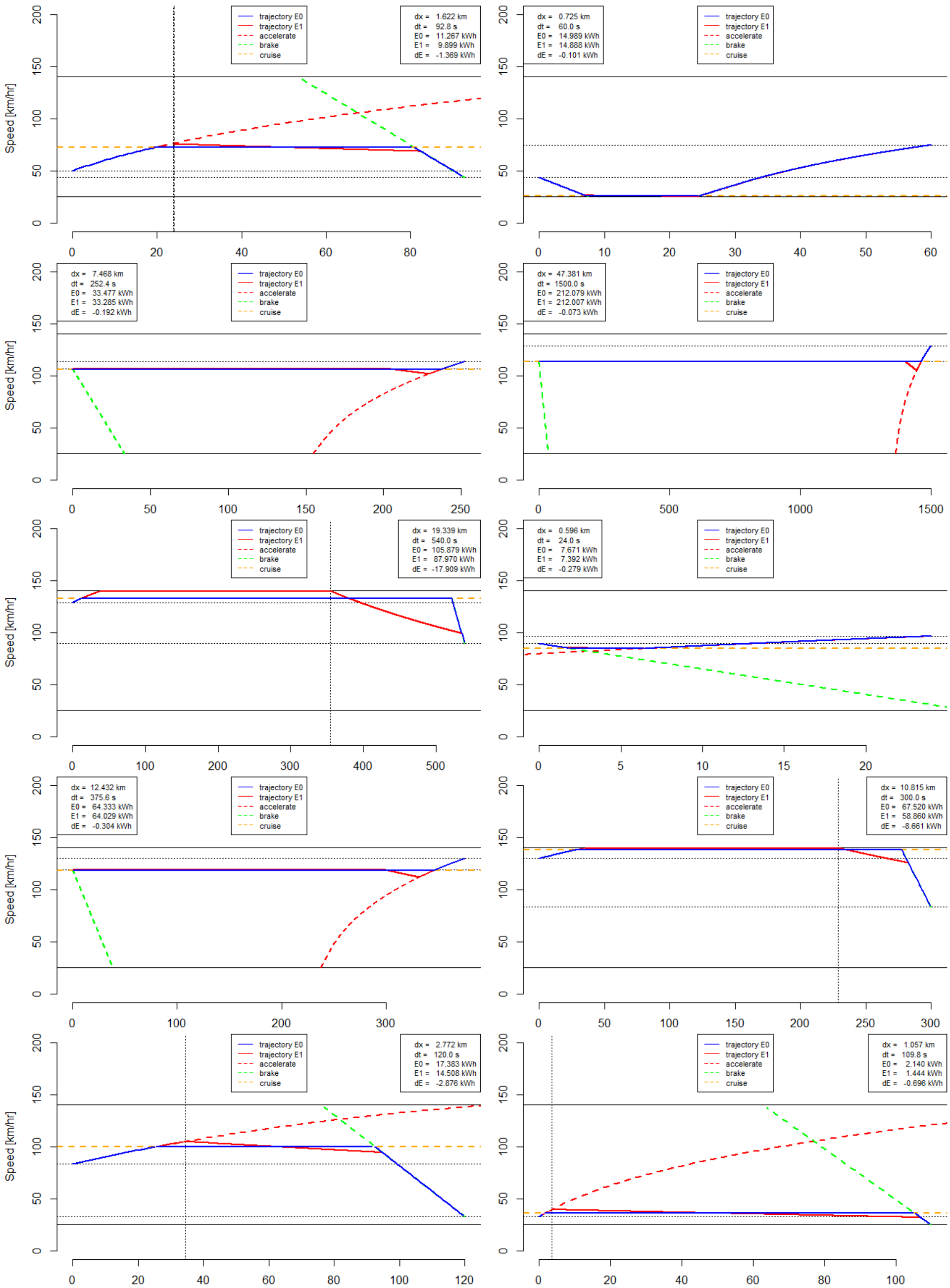

Figure 6: Heuristic solution for example track from Groningen to Zwolle. Each panel shows a segment of the journey. If it is not possible to accelerate enough during a segment (e.g. panel 2 in the top right), an additional acceleration phase is initiated after the segment, adjusting the next segment. These phases are not shown.

and $0 \le x_i \le 1$. Then the associated velocity profile $v(t)$ is a continuous function $v(t) \in C[0,1]$ that is restricted by the timetable with the following constraints:

$$v(0) = v(1) = 0 \text{ (full stop at terminal points);} \tag{11}$$
$$\int_0^1 v(t)\,\mathrm{d}t = 1 \text{ (total distance);}$$
$$\int_0^{t_i} v(t)\,\mathrm{d}t = x_i, \text{ for } i = 1,\dots,N \text{ (passing } x_i \text{ at time } t_i\text{);}$$
$$0 \le (t) \le v_{\max} \text{ and } a_{\min} \le v'(t) \le a_{\max} \text{ (velocity and acceleration limits) .}$$

The constraints do generally not determine $v(t)$ completely, allowing to search for the specific profile that realises the minimum of the energy functional

$$F(v) = \int_0^1 v[v' + \frac{r(v)}{\rho m}]^+ \,\mathrm{d}t, \tag{12}$$

where the nonlinear resistance $r(v)$ is defined in Eq. 2 according to the Davis model.

In order to apply a numerical optimisation algorithm we discretise the continuous function $v(t)$ by means of projection onto the space spanned by a convenient basis:

$$\tilde{v}(t) = \sum_{i=0}^{n} \alpha_i \phi_i(t),\ t \in [0,1].$$

For the sake of simplicity we demonstrate the concept for the piecewise-linear approximation on a uniform grid with step $h = \frac{1}{n}$. That is the approximation coefficients α_i are chosen so that

$$\tilde{v}(\frac{i}{n}) = v(\frac{i}{n}),\ i = 0,\dots,n,$$

and for $i = 0,\dots,n$ the basis functions are defined as

$$\phi_i(t) := \begin{cases} 1 - |nt - i|, & \text{if } |nt - i| \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

that have derivatives

$$\phi_i'(t) := \begin{cases} n, & \text{if } -1 \le nt - i < 0, \\ -n, & \text{if } 0 < nt - i \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

In this way, every $\phi_i(t)$ is supported only on interval $[i/n - h, i/n + h]$. Values of $\tilde{v}(t)$ and $\tilde{v}'(t)$ at grid points can be computed as a multiplication of the matrices M, D with coefficient column $\alpha = (\alpha_0,\dots,\alpha_n)^T$,

$$(M)_{i,j} = \phi_j(\frac{i}{n}),$$

$$(D)_{i,j} = \phi'_j(\frac{i}{n}).$$

The approximation to the energy functional (12) is now expressed as a function of α :

$$\tilde{F}(\alpha) = T\Big([D\alpha + r(\alpha)]^+ \cdot M\alpha\Big), \tag{13}$$

where functions $r(\alpha)$, $[\alpha]^+$ and multiplication $\cdot$ are taken element-wise and T implements appropriate integration quadrature. In the case of a linear basis this is the trapezoidal rule,

$$(T)_{i,j} = \begin{cases} \frac{1}{2(n-1)}, & 0 \le i - j \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, the cumulative integral of $v(t)$ is approximated by the vector product $q(\tau)^T\alpha$,

$$(q(\tau))_i = \int_0^\tau \phi_i(t)\mathrm{d}t.$$

Now, we are ready to formulate a non-linear optimisation problem that approximates the desired solution $v(t)$:
find a vector $\alpha \in \mathbb{R}^{n+1}$ such that

$$\begin{aligned} &(M\alpha)_0 = 0 \text{ and } (M\alpha)_n = 0; \\ &q(1)^T\alpha = 1; \\ &q(t_i)^T\alpha = x_i; \\ &0 \le (M\alpha) \le v_{\max}; \\ &a_{\min} \le (D\alpha) \le a_{\max}; \\ &\text{and put } \tilde{F}(\alpha) \to \min. \end{aligned} \tag{14}$$

To illustrate the concept let us consider the case when there is only one intermediate constraint, i.e., a train going from A to B has to pass intermediate point x_1 precisely a time t_1. We treat position as fixed, $x_1 = 0.5$, and by varying t_1 obtain a family of velocity profiles $v_{t_1}(t)$ corresponding to minimal energies, as shown in the left panel of Figure 7. One may observe that certain constraints yield optimal velocity profiles with lower energy cost than others, (see Figure 7, right panel). The velocity profile that has the smallest energy within the family is also the optimal velocity profile with no intermediate constraints. This observation can be used to adjust the given timetable in order to achieve even better energy efficiency (see Figure 8).

4.2 Optimisation of train timetable

Here, we assume that a train always travels according to the optimal velocity profile. The main question is: can we alter the existing set of constraints (i.e. timetable)

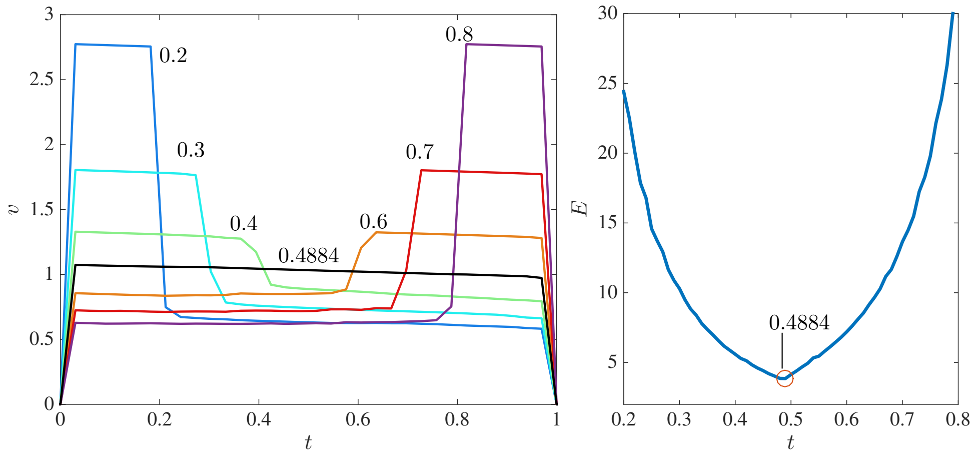

Figure 7: *L*eft: optimal velocity profiles for a single intermediate constraint with position $x = 0.5$ and various passing-time values (indicated). *R*ight: the optimal energy depends significantly on the passing time, t. The smallest optimal energy is reached if constraint's passing time coincides with the passing time of the unconstrainted velocity profile (i.e. 0.4884 for the current value of the constraint's position).

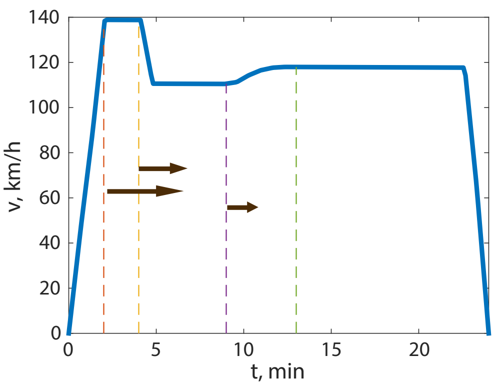

Figure 8: Optimal velocity profile for 4 constraints. The timetable can be improved by moving the constraints towards their optimal place (as if the constraint passing times belong to the unconstrained profile v_o.)

so that the energy consumption is even better? Small adjustments to the timetable (t_i, x_i) are feasible as long the timetable satisfies the periodic event scheduling model,

$$l_{i,j} \le (t_j - t_i) \mod 60 \le u_{i,j},$$

where t_i, t_j are event times and $l_{i,j},\ u_{i,j}$ are fixed limitations. In principle it is possible to directly set up an optimisation with an objective function defined as the energy of the timetable $f_\mathrm{o} = \tilde{F}(\alpha)$ where α solves the optimal velocity profile problem from the previous section. Such a routine, however, has to deal with a big non-linear optimisation problem and thus requires a good initial guess. We obtain this initial guess by running optimisation with a heuristic objective function. Let $v_\mathrm{o}(t)$ be an optimal energy profile with no intermediate constraints. We construct a heuristic objective function $f_\mathrm{h}(t_1, \ldots, t_N)$ that measure how far in L^2 norm is the given set of constraints t_i from passing times according to the optimal profile τ_i :

$$f_\mathrm{h}(t_1, \ldots, t_N) = \sum_{i=1}^{N} (t_i - \tau_i)^2,$$

where τ_i solves $\int\limits_0^{\tau_i} v_\mathrm{o}(t)\,\mathrm{d}t = x_i$. If a train makes stops at $(t_{x,i}, x_{s,i}),\ i = 1, \ldots, M$ we will additionally require the average speed between each pairs of stops be close to the overall average speed, v_avg (when calculated between terminal stations),

$$f_\mathrm{h}(t_1, \ldots, t_N) = \sum_{i=1}^{N} (t_i - \tau_i)^2 + \sum_{i=2}^{M} (t_{s,i} - (x_{s,i} - x_{s,j-1})/v_\mathrm{avg})^2. \tag{15}$$

Such an objective function provides a crude optimality estimate for a timetable. This estimate can be later used as an initial guess for, computationally more expensive, optimisation involving the functional $\tilde{F}(v)$ in 'predictor/corrector' combination. Table 1 depicts results of such an approach applied to a sample timetable. The first column of Table 1 contains information on the current timetable; the second column describes results of heuristic optimisation (CPU time less than 1 sec); the third column contains correction of the heuristic results by energy optimisation according to the functional $\tilde{F}(v)$ (CPU time 1.5 hour). Fragments of the optimal velocity profile for the optimised and original timetables are given in Figure 9.

Type	t_i	Predictor	Δ	Corrector	Δ
D	37	37	0	36.97	-0.03
P	41	41	0	40.99	-0.01
P	42	42	0	41.93	-0.07
A	44	44	0	43.96	-0.04
D	45	45	0	44.98	-0.02
P	47	47	0	47.00	0
P	48	48	0	48.01	$+0.01$

A	50	50	0	50.00	+0.01
D	51	51	0	50.98	−0.02
P	52	52	0	52.02	+0.02
P	54	54	0	53.94	−0.06
P	55	55	0	55.00	0
P	57	57	0	56.98	−0.02
P	58	58	0	57.96	−0.04
P	59	59	0	58.97	−0.03
P	2	2	0	2.00	0
P	6	5	−1	4.99	−1.01
A	7	6	−1	6.03	−0.97
D	8	7	−1	7.00	−1.00
P	13	13.23	+0.23	13.04	+0.04
P	14	14.23	+0.23	13.97	−0.03
A	21	21	0	21.00	0
D	23	23	0	22.97	−0.03
P	25	25	0	24.98	−0.02
P	27	27	0	26.93	−0.07
P	32	32	0	31.85	−0.15
P	36	36	0	35.83	−0.17
A	47	47	0	46.75	−0.25
D	48	48	0	47.67	−0.33
P	49	49	0	48.60	−0.40
P	50	50	0	49.65	−0.35
P	53	53	0	53.15	+0.15
A	58	58	0	58.07	+0.07
D	0	0	0	0.00	0
P	2	2	0	1.99	−0.01
P	3	3	0	3.00	0
P	4	4	0	3.98	−0.02
P	15	15	0	14.98	−0.02
P	20	19	−1	18.90	−1.10
A	24	23	−1	22.97	−1.03
D	26	25	−1	24.94	−1.06
P	37	35	−1	35.05	−1.95
P	38	36	−1	36.01	−1.99
A	45	46	+1.5	46.34	+1.34
D	41	36	−5	36.22	−4.78
P	43	43	0	43.02	+0.02
A	50	50	0	50.04	+0.04
D	53	52	−1	52.07	−0.93
P	4	4	0	4.01	+0.01
A	6	6	0	5.94	−0.06
D	10	10	0	9.93	−0.07

P	11	11	0	11.24	+0.24
P	23	26	+1	26.05	+3.05
A	25	28	+3	28.08	+3.08
Distance from original			24.0min		26.19min
Total energy			**87.39%**		**83.96%**

Table 1: A sample of a real timetable with 12 stops and 42 passing constraints. All distances are indicated in km and time in min. The timetable is consequently optimised with heuristic (predictor) and energy-functional (corrector) objective functions. The constraint types are encoded as follows: **D**eparture, **P**assing, **A**rrival. Distance from original indicates the sum of absolute changes in minutes.

4.3 Conclusions

For a given timetable we can find the optimal velocity profile numerically. This information may be presented to train drivers as an advisory. The routine computing optimal velocity profiles and energy is then further used to adjust the existing timetable. Such adjustment is done in two steps: heuristic objective function (cpu time 1sec, reduces energy down to 87.39 on sample data), and energy objective function (cpu time 1.5h, 83.96 on sample data). Even though the energy reduction is quite high, this approach involves numerical non-linear optimisation and does not necessarily lead to global minimum.

5 Conclusion and discussion

In this paper, we have looked at the problem proposed by NS. We considered two approaches. The first approach was primarily aimed at trying to reduce energy consumption while not changing the timetable. This was done by trying to understand what an optimal journey (with respect to energy consumption) looks like. Using this knowledge we developed a simple heuristic to optimise the usage of energy of a single train journey. This heuristic has been applied to a sample of actual train data and resulted in a energy reduction of 5%.

In the second approach, our aim was to compute for a given timetable the optimal energy profile numerically. Using this we applied numerical optimisation to a sample of an actual time table. Since the constraints Eq. 4 are modular this is not an easy task. However, taking the current timetable one can rewrite these constraint to absoute constraints. This resulted in a time table (for the sample) for which the optimal velocity profile yields a 16% energy reduction.

The main conclusion that can be drawn from this work is that energy consumption can in fact be reduced significantly. Not only by more efficient driving, but also by making small adjustments to the timetable allowing for more efficient velocity profiles. We note however that our results have only been applied to small samples of the timetable. To see what happens on a larger scale one should of course apply

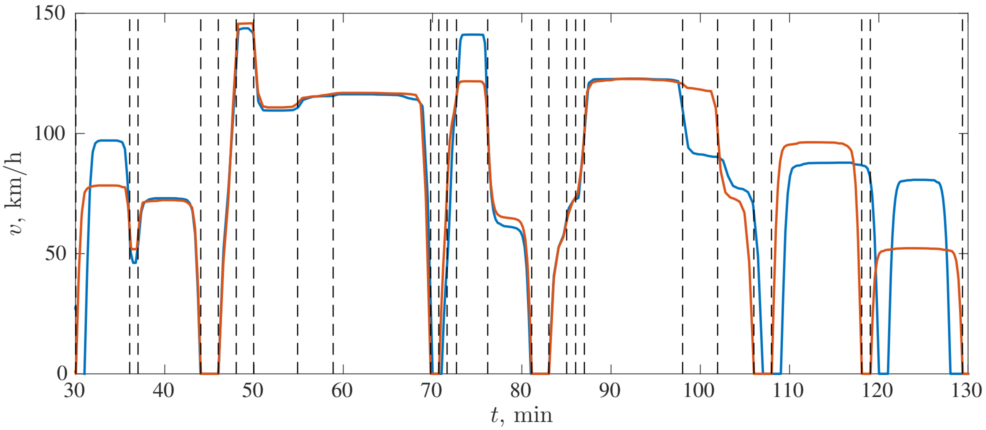

Figure 9: Fragments of optimal velocity profiles for current (*blue*) and improved (*red*) timetables. The vertical lines represent constraints after optimisation.

our results to the entire timetable. One thing that we observed is that prescribing time in minutes appear to make matters a bit complicated. For example the current timetable has some inconstancies, i.e. a train α should be at position x at time t but also on position x' at the same time. So it makes more sense to determine these times more accurately. Also from the point of view of energy reduction this makes sense. Allowing more flexible times values (not just entire minutes) can already lead to significant energy reduction (for the optimal profile).

It is not unlikely that the methods we have used can be improved. In particular, we believe that it would pay off to get a fast direct computation of the optimal velocity profile given a timetable. This could then be used to search for a better timetable with more advanced heuristics than we have currently employed.

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