Always Nice Weather in Europe

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Abstract

Weather forecasting relies on mathematical models that exhibit chaotic behavior. This renders the solution of these models very sensitive to errors in the model, to choices of the initial conditions and to truncation errors in the numerical solution procedure. Over the course of the past decade, various meteorological institutes in Europe have developed different atmosphere models. Each of these models has its strengths and weaknesses. The principle behind the so-called Super Modeling approach is to merge these existing models into a single larger model to combine common strengths while overcoming individual weaknesses. This approach was initially proposed and developed by the KNMI in the Netherlands to improve the reliability of its weather forecasts. The task formulated for this Study Group problem was to reevaluate the Super Modeling approach and to formulate recommendations for its future development.

1 Introduction

Meteorological institutes are continually seeking to improving their weather forecasts. Research is directed towards minimizing the discrepancy between mathematical models predicting atmospherical conditions and actual measurements. These mathematical models typically require the solution of systems of non-linear ordinary differential equations that allow for solutions with a chaotic behavior. Thus any solution procedure is prone to errors in the modeling, in the initial conditions and in the numerical time integration procedure. This has lead to the development of different models by various European institutes with their own strengths and weaknesses. None of these models is currently accredited of giving the best overall simulation results.

To overcome the absence of a universally best weather simulation environment, the Dutch meteorological institute KNMI in collaboration with partner institutes pioneered an approach in which different models are combined into a larger model. This combination aims at exploiting the strength of different approaches while overcoming individual weaknesses. Weather models are combined by synchronizing their outputs, i.e., by penalizing the deviation of a single model from a common prediction of a time evolution. The guiding principle is that the common realization agrees better with reality than each of the individual ones. The underlying idea of *synchronization* [1] is known to play an important role in e.g. social sciences (e.g. in the common start and ending of applause for a performance) and in biology (e.g. in the migration of flocks). The KNMI and partners coined their approach the *Super Modeling* (SUMO) approach [2].

Given its recent development, the SUMO approach generates a large number of interesting research questions. Examples include:

- weather forecasting models describe the time evolution of an n-dimensional state vector. It is not a-priori clear how many and which components of this state vector should be coupled in order to obtain synchronization. Neither is clear whether this set should change in time. A mechanism to enforce synchronization using a small (and possibly time-dependent) set of components would be beneficial to have;
- in their studies, the KNMI and partners enforced synchronization by constraining the norm of the difference between state vectors corresponding to the individual models. This mechanism is referred to as *linear nudging*. It is not a-priori clear whether this mechanism is optimal and what alternative synchronization mechanism should be considered;
- it not immediately clear what value the coupling coefficient in a linear nudging technique should have and what strategies could be developed to obtain these values by matching with previously recorded data as in see [3];
- it is intuitively clear that a very weak coupling of M individual models results in a super model exhibiting the M individual dynamic behaviors. A very strong coupling on the other hand results in a super model with

a single dynamic that is somehow the average of the individual dynamics leading to incorrect results. The idea is that intermediate coupling strengths are most appropriate. However, KNMI and partners have observed that for certain choices of intermediate coupling strength a system with new dynamics arise. So-called *ghost-attractors* arise with predicted outcome: *always nice weather in Europe*. The question was to find an explanation for this.

The Study Group was asked to consider the above issues. The Lorenz-63 model will be used as an illustrative example and guide in the numerical studies.

This report is structured as follows. This Introduction will be followed by three main sections. In Section 2 we consider a technique based on coupling restricted to unstable directions in the tangent space. In Section 3 we consider the coupling of three copies of the Lorenz model. In Section 4 we study the appearance of ghost attractors. In Section 5 we study alternative coupling approaches. Conclusions finally are drawn in Section 6.

2 Dynamical Properties of Imperfect Models and the Supermodel

The true state of a physical system is assumed to be given by a set of observations $\{t_i, P(t_i)\}$, where $P(t)$ denotes the state vector of the truth system at time t. Available are a set of imperfect models for which the values of parameters can be obtained by fitting the model to the observations. The study focuses on systems with chaotic dynamics, which are described as follows by the system of differential equations

$$\frac{dx}{dt} = f(x) \tag{1}$$

in a n-dimensional state space. Let $x(t) = p(t)$ be a chaotic solution, then other trajectories nearby this chaotic orbit are analyzed by making use of the tangent linear system. By substituting

$$x(t) = p(t) + v(t)$$

into (1) and preserving only the linear terms one obtains the tangent linear system

$$\frac{dv}{dt} = F[p(t)]\, v(t) \text{ with } F[p(t)] = \{\partial f_i(p(t))/\partial x_j\}_{n\times n} \tag{2}$$

in which $v(t)$ denotes the perturbation from $p(t)$. The matrix F has time-dependent eigenvalues. Their averages over a full orbit are called the Lyapunov exponents. For dissipative systems the sum of these exponents is negative. The orbit $p(t)$ is chaotic if at least one of the exponents is positive. It means that in certain parts of the state space at least one eigenvalue must be positive and that an orbit must pass such a region from time to time in order to have a chaotic orbit.

The Lorenz-63 model is given by

$$\frac{dx_1}{dt} = \sigma(x_2 - x_1), \tag{3}$$

$$\frac{dx_2}{dt} = x_1(\rho - x_3) - x_2, \tag{4}$$

$$\frac{dx_3}{dt} = x_1 x_2 - \beta x_3. \tag{5}$$

From the Lorenz-63 system it is known that in the 3D state space near the origin such a region exists. There the velocity of the trajectories has a large component in the direction of the x_3-axis, so a bundle of trajectories moves in a direction of x_3 and exhibits after passage of the origin a strong tendency to diverge. This means that, if one wants to perturb a chaotic orbit $p(t)$, one has to do that in a direction perpendicular to the x_3-axis. In order to synchronize two imperfect (Lorenz-63) models, which both pass in a similar manner the region near the origin, one has to apply a coupling of the form

$$\frac{dx_1}{dt} = f_1(x) + c_1(y_1 - x_1) \qquad \frac{dy_1}{dt} = g_1(y) + c_1(x_1 - y_1) \tag{6}$$

$$\frac{dx_2}{dt} = f_2(x) + c_2(y_2 - x_2) \qquad \frac{dy_2}{dt} = g_2(y) + c_2(x_2 - y_2) \tag{7}$$

$$\frac{dx_3}{dt} = f_3(x) + c_3(y_3 - x_3) \qquad \frac{dy_3}{dt} = g_3(y) + c_3(x_3 - y_3) \tag{8}$$

where $f(x)$ and $g(y)$ may differ in just the values of the model parameters. Thus, for the Lorenz-63 system any vector $c = (c_1, c_2, 0)$ may give rise to synchronisation for the two coupled imperfect models. Also the size of the coupling vector c may play a role. The best coupling vector is found by composing the *super model*

$$S(t; c) = [x(t; c) + y(t; c)]/2 \tag{9}$$

and find the best fit of this model to the available observations $\{t_i, P(t_i)\}$. For higher dimensional systems such computations may need too much time.

Then one may concentrate on the eigenvector that corresponds with the largest (positive) eigenvalue. Together with the eigenvector, that corresponds with the eigenvalue equal to zero, it spans the *most unstable* manifold of the chaotic orbit, see Figure 1. In the direction of the flow given by the vector dp/dt a perturbation $v(t)$ is neither damped nor does it tend to explode, so in that direction the system is neutrally stable and yields therefore an eigenvalue equal to zero. Thus, one may take the eigenvector that corresponds with the largest eigenvalue as the coupling vector c with the appropriate length or choose its projection in the space perpendicular to dp/dt.

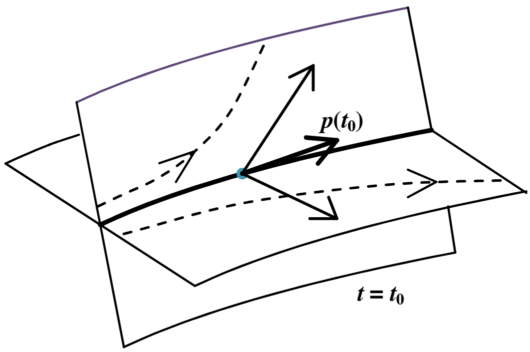

Figure 1: Dynamics of the trajectories near the chaotic orbit, being a stable strange attractor. There the 3D state space can be decomposed in two manifolds together with the vector $p(t)$. Near a point $p(t)$ on the attractor these manifolds are planes which are spanned by the eigenvector $p(t)$ and each of the two other eigenvectors. Depicted is the case that the two manifolds are unstable. For chaos it suffices that only one of the manifolds is unstable.

3 First numerical exploration of a coupled Lorenz-63 system

In this section, we study the following system of two identical Lorenz models with different initial data and linear coupling:

$$\begin{aligned}
\dot{x}_1 &= \sigma(y_1 - x_1) \ + \ C_1^x(x_2 - x_1) \\
\dot{y}_1 &= x_1(\rho - z_1) - y_1 \\
\dot{z}_1 &= x_1 y_1 - \beta z_1 \\
\dot{x}_2 &= \sigma(y_2 - x_2) \ + \ C_2^x(x_1 - x_2) \\
\dot{y}_2 &= x_2(\rho - z_2) - y_2 \\
\dot{z}_2 &= x_2 y_2 - \beta z_2
\end{aligned} \tag{10}$$

where we choose $\sigma = 10, \rho = 28$ and $\beta = \frac{8}{3}$. This is a typical choice of parameters that is used often for the Lorenz system; chaos is found and the famous butterfly attractor arises.

It has been shown recently, see [2], that limited information exchange between two identical Lorenz systems can lead to synchronization of the model states even when the systems are initialized from very different initial conditions and differ slightly in parameter values. The ability to synchronize with the truth measures the quality of the model.

Through trial and error, it has been found that often just a part of the state space vectors need to be exchanged between the models in order for the models to synchronize on a common solution.

A first question is whether the coupling should be symmetric in the phase space variables. For us *symmetric coupling* for a system means that the coupling constant for one variable in the first model must be the same as the coupling constant for the same variable in the second model. In terms of the system (10) it means that $C_1^x = C_2^x$. For three systems with different parameters it was demonstrated in [2] that couplings need not be symmetric.

Indeed, we could experimentally show that the solutions of two identical Lorenz models with different initial condition need to be coupled only in the x-variables, and the coupling does not have to be of the same strength in both equations. For the above coupled model (10), a total coupling strength of at least 9 was necessary, and synchronisation happened both when this coupling strength was imposed in only in one of the models, so either $C_1^x > 9$ and $C_2^x = 0$, or $C_1^x = 0$ and $C_2^x > 9$, or when the coupling strength was achieved cumulatively, i.e. with $C_1^x + C_2^x > 9$

An important question is: given two identical Lorenz Models with different initial conditions, which are the coupling strengths that lead to synchronization without essentially changing the dynamics?

Using a straight-forward implementation of two identical Lorenz models in Mathematica, we verified the following values: Synchronization happens for a cumulative coupling strength of above 9, with the critical values being $C_1^x = 8$ and $C_2^x = 1$, where a long time has to pass until the solutions synchronize: Only after $t = 60$ do we observe a sufficiently small error in the x-coordinates, which is then still several orders of magnitude bigger than in the symmetric case. For example, for $C_1^x = 10 = C_2^x$, the error falls below 10^{-6} for $t > 20$.

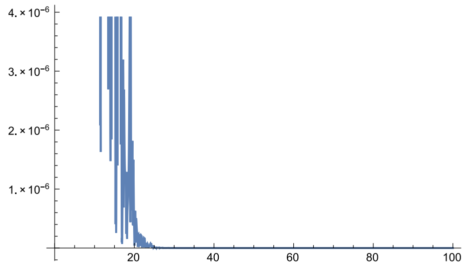

Figure 2: Error for $C_1^x = 10 = C_2^x$

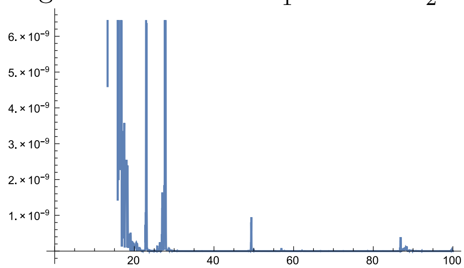

Figure 3: Error for $C_1^x = 20$ and $C_2^x = 10$

In terms of an upper bound on the coupling constants, we observed an unexpected robustness of the dynamics: Even for quite high values like $C_1^x = 20$ and $C_2^x = 10$, synchronization seems to happen for times $t > 30$, but sudden spikes in the error $|x_1(t) - x_2(t)|$ of order 10^{-6} keep appearing in irregular intervals.

These results are stable and not sensitive even to significant changes of the initial conditions (i.e. of values of ± 20 per phase space coordinate).

We observed experimentally that the achieved synchronization is lost even when the parameters of the Lorenz models are changed slightly. For example, choosing $\beta = 3$ instead of $\beta = \frac{8}{3}$ in one model destroys the effect completely.

4 Stability analysis of ghost attractors

4.1 Introduction

It has been observed [4] that coupling multiple models to form a super model may lead to the super model getting “stuck" in a certain part of phase space (such that it is always nice weather in Europe). In the context of a number of coupled Lorenz models, this can lead to all models being fixed at two points close to the unstable fixed points in the Lorenz butterfly [4]. In this case some models will be in a point on one wing, some others will lie on the other wing. The precise position of the points where they are fixed depends on the exact formulation of the super model.

We analyse this behaviour by considering a simpler model in which we couple identical models. We then analyse the fixed points and their stability. In case a fixed point is stable we can conclude that there is a set of initial conditions for which the supermodel converges to this fixed point solution.

4.2 The model and our simplifying ansatz

We consider $N+M$ coupled Lorenz63 models all with the same choice of the coefficients, with linear and uniform coupling and standard parameters, leading to the following equations (with $i = 1, 2, ..., N+M$)

$$\begin{aligned}
\dot{x}_i &= 10(y_i - x_i) \ + \ C \sum_{j=1}^{N+M} (x_j - x_i) \\
\dot{y}_i &= x_i(28 - z_i) - y_i \ + \ C \sum_{j=1}^{N+M} (y_j - y_i) \\
\dot{z}_i &= x_i y_i - \frac{8}{3} z_i \ + \ C \sum_{j=1}^{N+M} (y_j - y_i).
\end{aligned} \tag{11}$$

Since we know from [4] that these oscillators may get stuck into a situation in which there are N oscillators in some fixed point and M other oscillators also together in one (possibly different) point, our approach now consists of finding all possible fixed points under the assumption that the oscillators are stuck in at most two points, for given C, N and M. If we label one fixed point by $(x_*^{(1)}, y_*^{(1)}, z_*^{(1)})$ and assume that M oscillators are stuck in this point, and the other by $(x_*^{(2)}, y_*^{(2)}, z_*^{(2)})$ where N oscillators are stuck. Then the equations simplify to

$$\begin{aligned} \dot{x}_i &= 10(y_i - x_i) \; + \; CM(x_*^{(1)} - x_i) \; + \; CN(x_*^{(2)} - x_i) \\ \dot{y}_i &= x_i(28 - z_i) - y_i \; + \; CM(y_*^{(1)} - y_i) \; + \; CN(y_*^{(2)} - y_i) \\ \dot{z}_i &= x_i y_i - \frac{8}{3} z_i \; + \; CM(z_*^{(1)} - z_i) \; + \; CN(z_*^{(2)} - z_i). \end{aligned} \tag{12}$$

An important remark is that these equations are only consistent in case the oscillators lie actually exactly in these points. Hence, at these fixed points our system of equations reduces to M copies of

$$\begin{aligned} 0 = \dot{x}_*^{(1)} &= 10(y_*^{(1)} - x_*^{(1)}) \; + \; CN(x_*^{(2)} - x_*^{(1)}) \\ 0 = \dot{y}_*^{(1)} &= x_*^{(1)}(28 - z_*^{(1)}) - y_*^{(1)} \; + \; CN(y_*^{(2)} - y_*^{(1)}) \\ 0 = \dot{z}_*^{(1)} &= x_*^{(1)} y_*^{(1)} - \frac{8}{3} z_*^{(1)} \; + \; CN(z_*^{(2)} - z_*^{(1)}) \end{aligned} \tag{13}$$

and N copies of

$$\begin{aligned} 0 = \dot{x}_*^{(2)} &= 10(y_*^{(2)} - x_*^{(2)}) \; + \; CM(x_*^{(1)} - x_*^{(2)}) \\ 0 = \dot{y}_*^{(2)} &= x_*^{(2)}(28 - z_*^{(2)}) - y_*^{(2)} \; + \; CM(y_*^{(1)} - y_*^{(2)}) \\ 0 = \dot{z}_*^{(2)} &= x_*^{(2)} y_*^{(2)} - \frac{8}{3} z_*^{(2)} \; + \; CM(z_*^{(1)} - z_*^{(2)}). \end{aligned} \tag{14}$$

We can now consider the system consisting of (one copy of each of) (13) and (14). The solutions of this combined system and the Jacobian at these fixed points can be found using Mathematica. From the eigenvalues of this Jacobian conclusions about the stability of the fixed points van be determined.

4.3 Stability analysis

We did the stability analysis for (13) and (14) for the parameter values $C = 0.6$, $N = 8$ and $M = 12$ and also for $C = 0.2$, $N = 1$ and $M = 19$.

For the first parameter set, $C = 0.6$, $N = 8$ and $M = 12$, we find 9 fixed points, of which 2 are stable. The fixed points can be categorized as follows: 3 unstable fixed points at the unstable fixed points of the original single system (i.e. at the origin and wings)[1]. There are also emergent fixed points that arise through the interactions between the oscillators. These can be separated into a pair of stable fixed points on the wings and two pairs of unstable fixed points with one group of oscillators on a wing and another group close to the origin. The stable fixed points on the wings also emerged in the simulations done in [4]. In figure 4 we depict all fixed points, together with a sample trajectory of one Lorenz63 model for reference.

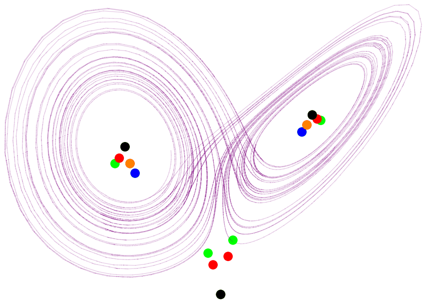

Figure 4: The fixed points for the parameters $C = 0.6$, $N = 8$ and $M = 12$. The single system fixed points are in black, the unstable emergent fixed points where N oscillators can be are green, the unstable emergent fixed points where M oscillators can be are red, the stable emergent fixed points where N oscillators can be are blue, the unstable emergent fixed points where M oscillators can be are orange and a sample trajectory of one Lorenz63 model is depicted in purple.

For the second parameter set, $C = 0.2$, $N = 1$ and $M = 19$, we find only 5 fixed points, none of which are stable. We retain the fixed points of the single system (which should not be too surprising) and we also retain two emergent unstable fixed points, where there are 19 oscillators close to the origin and

[1] This corresponds to all systems being synchronized at an identical fixed point, such that the couplings terms are all zero.

one oscillator in a wing. Since there are no stable fixed points here, it is not surprising that configurations in which 19 oscillators balance out one have not been observed in the simulations [4]. In figure 5 we again depict all fixed points and a sample trajectory of one Lorenz63 model for reference.

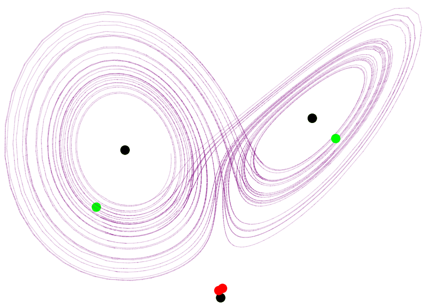

Figure 5: The fixed points for the parameters $C = 0.2$, $N = 1$ and $M = 19$. The single system fixed points are in black, the unstable emergent fixed points where N oscillators can be are green, the unstable emergent fixed points where M oscillators can be are red and a sample trajectory of one Lorenz63 model is depicted in purple.

Our analysis can in principle be applied to any values of C, N and M in order to find any possible ghost attractors. We have found that for more balanced values of N and M there are more fixed points and they may be stable. We observed that for very skewed distributions (all stable and some unstable) fixed points disappear. This analysis could possibly be extended to more complicated configurations of oscillators than a division into two groups. An extension to more complicated models could also be of practical use. An interesting open question that remains is how the unstable emergent fixed points influence the dynamics of the coupled system.

a

b.

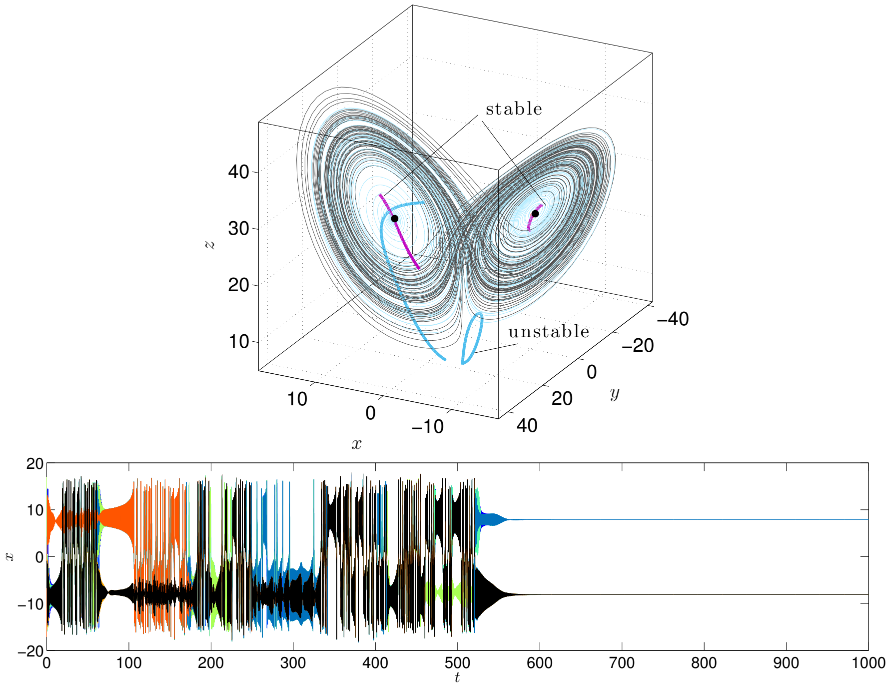

Figure 6: Simulation results for a Lorentz '63 based supermodel. *a.* Integration lines of the synchronised supermodel and two families of critical points that were obtained for various values of coupling strength C: stable (magenta) and unstable (blue). *b.* Time dynamics of x-variables depicts the synchronisation process leading to a steady state solution that is not featured by the original Lorentz '63 model.

5 A search for improved coupling mechanisms

Let a supermodel

$$\dot{x}_{i,j}(t) = L_i(x_{i,j}(t)) - \sum_{k \neq j} C_{j,k}(x_{i,j}(t) - x_{i,k}(t)),\ i = 1, \ldots, m,\ j = 1, \ldots, n; \tag{15}$$

be composed of n instances of autonomous basic models,

$$\dot{x}_i(t) = L_i(x_i(t)),\ i = 1, \ldots, m. \tag{16}$$

Here $C_{j,k} > 0$, are constants, that define the strength of connection for each couple of models. It is known that in some cases the dynamics of the synchro-

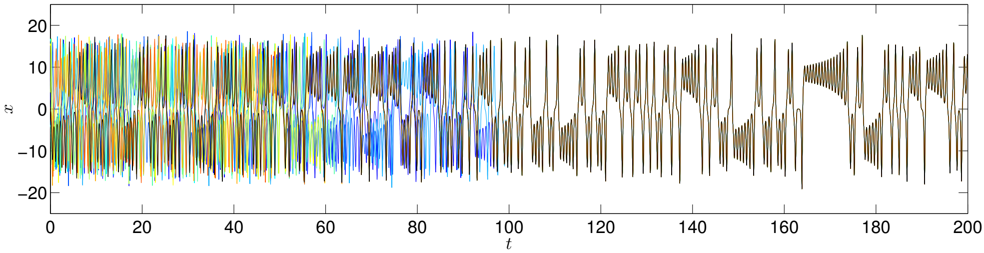

Figure 7: Time dynamics of x-variables of Lorenz '63 based supermodel with Gaussian kernel coupling, The synchronisation process leads to a chaotic but synchronous solution

nised supermodel (15) may be different from the dynamics of the basic model (16). For instance, numerical analysis show extra stable critical points appear in Lorentz'63 based supermodel as in Figure 6. One possible way to overcome the issue is to define the coupling mechanism to be active only locally, when a pair of submodels are close enough in the state space. This can be done by redefining (15) as

$$\dot{x}_{i,j}(t) = L_i(x_{i,j}(t)) - \sum_{k\neq j} C_{j,k}\phi_{j,k}(|x_{i,j}(t) - x_{i,k}(t)|)(x_{i,j}(t) - x_{i,k}(t)), \quad i = 1,\ldots,m,\ j = 1,\ldots,n; \tag{17}$$

where $\phi(x),\ x \geq 0$ is a smooth function with a maximum at $x = 0$ that is finite supported or decays on infinity as $o(x^{-1})$. A good candidate for $\phi(x)$ is a Gaussian kernel, $\phi(x) = e^{-x^2/\sigma^2_{j,k}}$. Numerical simulations for a supermodel containing 10 identical Lorenz'63 models coupled with the Gaussian mechanism (17) reveal that although synchronisation process takes more time, the synchronised solution is not a steady state one, as can be seen in Figure 7. Note, the price to pay for using this approach is an extra parameter $\sigma_{j,k}$ that together with $C_{j,k}$ should be estimated by an optimisation/machine learning process.

6 Conclusions

Meteorological institutes are keen to increase the reliability of their weather forecasts. They are faced with the challenge that the underlying mathematical models exhibit chaotic behavour and are therefore hard to analyze and solve. To overcome this challenge, the KNMI and partners have developed the super-modeling approach in which a set of models with different strengths and weaknesses are coupled. We were asked to look into this approach and formulate recommendations for its future development. We found that coupling the state space variables in the hyper plane perpendicular to the orbit can be sufficient to obtain synchronization of the different models. The new dynamics of the super model was briefly looked into. Coupling mechanisms that differ from linear nudging were studied and coupling by Gaussians was found to be effective in particular circumstances. Overall, more research is required to obtain a better understanding of the super modeling approach to obtain more reliable weather forecasts.

References

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