Nonlinear Cochlear Dynamics

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Abstract

In this report we examine a model for human hearing. The unknown parameters in the model are estimated using experimental data and standard optimisation methods as described in the text. Additionally, we suggest possible improvements to the model as well as proposing a method to use the current model in locating which frequencies are affected in a damaged ear.

KEYWORDS: cochlear model, delay differential equation, parameter estimation, hearing loss

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1 Introduction

INCAS[3] posed the problem of modeling the human hearing system in the Study Group of Mathematics with Industry held at TUDelft. More specifically, the company brought to our attention a model describing the part of the ear called cochlea as a series of coupled oscillators. Each oscillator is modelled by a second order linear ordinary differential equation with a delay term. The questions that mainly concerned us were the following:

1. Is it possible to improve this model?
2. Is it possible to estimate the parameters of this model using experimental data?

In the course of four days, we attempted to answer these questions as accurately as possible. In addition, the company was interested in a mathematical description of how a damaged ear works in comparison to a healthy one. While this proved to be an impossible task, we were able to suggest a method for locating the frequencies that are affected in a damaged ear.

This article is organised as follows: in section 2, we present the model used and the theoretical background it is based upon. In section 3, we analyse the model and attempt a physical interpretation of it. Section 4 deals with the mathematics of hearing loss diagnosis, while in section 5 we propose methods to estimate the parameters of the model. Finally, we present conclusions of our investigations as well as future directions of research.

2 Theoretical background

The model presented in this section is described in more detail in a paper of Zweig (1991).

Before analyzing the model that describes how the human ear works, let us first consider the anatomy of the human ear and, more specifically, the cochlea. It has been known for quite some time that the cochlea is a nonlinear, active system that converts sound into neural stimuli. In addition, the cochlea not only responds to the sound it receives, but emits sound as well. These OtoAcoustic Emissions (OAEs) can be accurately measured however, due to the nonlinearity of the cochlea, their use in revealing how the cochlea responds to certain (controlled or not) stimuli is non-trivial.

In his paper, Zweig considers a simplified model of the human ear by 'uncoiling' the cochlea, a model already existent in the literature. He notes that discrepancies appear between this theory and some experiments, which might be caused by deficiencies of the model, namely the possibility of oversimplifying what actually happens in the cochlea. However, relaxing the assumptions

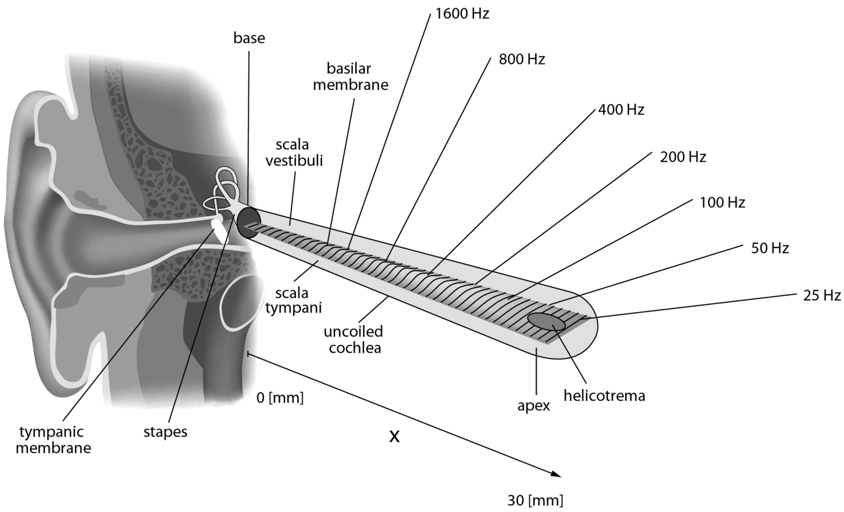

Figure 1: *Simplified model of the inner ear with uncoiled cochlea. Reproduced from* `wikimedia.com`

the model uses does not lead to any improvements. One of the assumptions, for instance, is that the geometry of the cochlear model is excessively simplified by the uncoiling of the cochlea. In addition, the fluids in the scalae are considered incompressible and inviscid. Relaxing these assumptions, i.e. assuming the cochlea is coiled and allowing the fluid to be compressible and viscous, only slightly changes the output of the model, which still does not fit the experimental data. Furthermore, the author questions the assumption that the scala media should be interpreted as an array of oscillators, coupled only through the fluid inside the ear. However, adding additional coupling between adjacent oscillators also fails to improve the output of the model.

Instead of relaxing the initial assumptions, the author uses a different approach. He considers the initial model as correct in a number of cases but acknowledges it is too simple to fully capture the correct behavior. Then, he proceeds to replace the harmonic oscillators of the model with more complex oscillators. Using the data available it is possible to approximate the form of the refined transport function and use it to obtain the more complex oscillator equation. We will now describe this approach in more detail, as there are parts of this procedure that could be altered, possibly resulting in a further improvement.

To get an oscillator equation from the transport equation, let us consider the latter as an integral equation for λ and suppose that T can be obtained from experimental data:

$$T(s) \simeq C \frac{s}{\lambda^{3/2}(s)} \exp\left(-\int_{s_0}^{s} \frac{ds'}{\lambda(s')}\right) \tag{1}$$

We can now get the oscillator equation by Fourier transforming $\lambda^2 V \propto sP$ using the form of λ from the simple harmonic oscillators model:

$$\lambda^2 = \frac{s^2 + \delta s + 1}{(4N)^2},$$

where N is approximately equal to the number of wavelengths of the wave on the membrane. This will give us an inhomogeneous oscillator equation for the velocity v of a point on the basilar membrane, where $v = \mathcal{F}(V)$ and $\mathcal{F}$ denotes the Fourier transform, namely

$$\ddot{v}(\theta) + \delta \dot{v}(\theta) + v(\theta) = \frac{\dot{p}(\theta)}{\omega_{c0} M_0},$$

where $\theta(x) = \omega_c(x)t$ and $(\,\dot{}\,) = \partial/\partial\theta(x)$. We can now differentiate the transfer equation to solve for λ:

$$\lambda = -\left(1 + \frac{3}{2}\frac{d\lambda}{ds}\right)\left(\frac{d\ln(T/s)}{ds}\right)^{-1}. \tag{2}$$

Because the derivative of λ is small, we can get an approximation for λ as

$$\lambda \simeq -\frac{ds}{d\ln(T/s)}. \tag{3}$$

As λ is heavily influenced by the derivative of T, it is important to get an approximation for T as smooth as possible. To this end, a data fitting is performed by maximizing a modified likelihood function of the form $\chi^2 + \xi k^2$ where

$$k^2 = \int \left|\frac{T(s(\Omega))}{d\Omega^2}\right| d\Omega$$

and ξ plays the role of a Lagrange multiplier. This form of likelihood function is preferred over the usual χ^2 as s depends on a number of variables and this will cause T to be non uniformly distributed with respect to χ^2.

Let us now assume that the linear equation for the shunt impedance Z is correct but incomplete because it fails to capture all the mechanical properties of the cochlea. To correct it, we add an extra term that will account for these mechanical properties. The equation now is

$$Z = \frac{\omega_{c0} M_0}{s}\left(s^2 + \delta s + 1 + m(s)\right)$$

which can also be expressed as an equation for λ:

$$\lambda^2 = \frac{\left(s^2 + \delta s + 1 + m(s)\right)}{(4N)^2}.$$

To gain some insight into the form of the function $m(s)$, we fix N and δ and iteratively calculate λ from equations (2) and (3). We then plot the imaginary versus the real part of $m(s)$ for varying s. The author approximates the resulting points by a circle of the form $m(s) = \rho e^{-2\pi\mu s}$, ρ and μ being real constants. Substituting into the equation for the shunt impedance gives

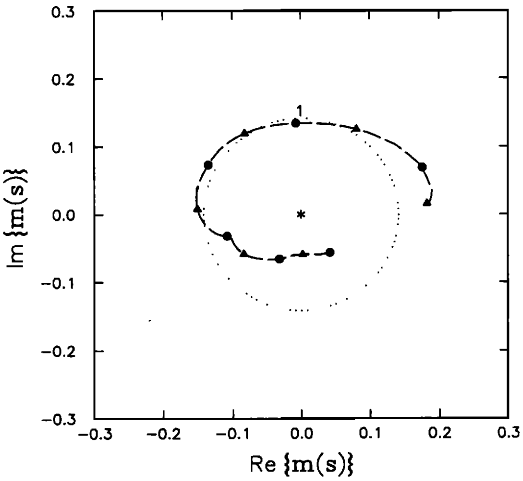

Figure 2: *The dashed curve is the measured values of $m(s)$ obtained from the transfer function T whereas the doted line is a circular approximation. The dots represent an equally spaced partition of the frequency domain.*

$$Z = \frac{\omega_{c0} M_0}{s} \left(s^2 + \delta s + 1 + \rho e^{-2\pi\mu s}\right).$$

Now using the Fourier transform as above yields the new oscillator equation for the velocity v of a point:

$$\ddot{v}(\theta) + \delta\dot{v}(\theta) + v(\theta) = \frac{\dot{p}(\theta)}{\omega_{c0} M_0} - \rho v(\theta - \psi),$$

where $\psi = 2\pi\mu$. Therefore, a section of the organ of Corti at position x behaves like a harmonic oscillator with angular frequency 1 and damping δ. It is also driven by two forces: one proportional to the derivative of the pressure difference and one proportional to that section's velocity at the earlier time $\theta - \psi$. This delayed force is necessary to stabilize an otherwise unstable oscillator (recall that the damping is negative) and can be considered the active influence of the cochlea. An estimate of the time delay is given by

$$\frac{\psi}{\omega_c(x)} = 220\mu s,$$

where an approximation for μ (and therefore ψ as well) was obtained by fitting the model to experimental data. Zweig's model undoubtedly has a number of advantages. It describes the physical phenomenon much better than the simple harmonic oscillator model while at the same time remaining relatively simple for analysis and solution. However, an obvious point of improvement lies in the approximation of the form of the unknown function m by a circle. It is clear from the plot that an ellipsoid or a spiral would better fit the experimental data and result in a more accurate oscillator equation, without changing the assumptions of the original model or excessively complicating the calculations involved. We strongly believe that this point needs to be examined and reassessed in the future.

3 Physical interpretation

The processes involved in the perception of sound from a pressure wave are numerous. The most important process occurs in the cochlea, which makes the crucial transformation of a pressure wave into an electric signal, which can then be interpreted by the brain as sound. This transformation is a two-step process. First a small membrane inside the cochlea is made to oscillate by a propagating pressure wave. Then, the oscillation is registered by hair cells that activate the firing of an electric signal.

This oscillatory behavior of the cochlea was modeled by considering the cochlea as a tubular resonance cavity which encloses a membrane of oscillators that lies on the central horizontal plane. This plane divides the resonance cavity in two cavities, which are only connected at the far end of the cochlea. Research has shown that the oscillators responding to a pressure wave of a certain frequency have a position on the membrane which increases with decreasing frequency. In other words, higher frequencies are processed near the outer part of the cochlea and lower frequencies near the inner part. As we saw above, this frequency dependency of position was modeled with a delay differential equation. The model assumed a one-dimensional position x with domain $[0, 1]$, a total transversal pressure $p(x, t)$ and a transversal displacement $\xi(x, t)$ of the oscillators. The pressure p is actually the difference in pressure between the two cavities in the cochlea. The delay differential equation in the paper is written in a slightly more general form as

$$p = m\ddot{\psi} + d\dot{\psi} + s\psi + s'\psi_{t-\tau}, \tag{4}$$

where $\psi_{t-\tau} = \psi(t-\tau)$ for some specific time $\tau > 0$. The differential equation without delay is that of a driven harmonic oscillator. There is extensive theory about this type of differential equation that leads us to expect certain parameter dependencies. INCAS[3] provided the following parameter dependencies

$$d = c_0\sqrt{sm}, \quad s' = c_1 s, \quad \tau = c_2\sqrt{\frac{m}{s}}, \tag{5}$$

where c_0, c_1 and c_2 are dimensionless constants.

Using these parameter dependencies we can transform equation (4) into a dimensional differential delay equation

$$\mathcal{P}(x,t) = \ddot{\xi}(x,t) + \frac{c_0 c_2}{\tau}\dot{\xi}(x,t) + \left(\frac{c_2}{\tau}\right)^2 \xi(x,t) + c_1\left(\frac{c_2}{\tau}\right)^2 \xi(x,t-\tau) \tag{6}$$

where a dot denotes partial differentiation with respect to time and $\mathcal{P}$ is defined as p/m, which has the same dimensions as acceleration.

Equation (6) fails to fully describe the ongoing process. All the oscillators are behaving independently and the pressure is known only at $x = 0$, since we only know the sound that enters the ear canal. The missing link are the cavities, which allow the existence of standing waves. However the Navier-Stokes equation for an incompressible, inviscid cochlear fluid with pressure differences only implies that an oscillator can influence these standing pressure waves. Therefore the Laplacian equation of standing waves becomes a Poisson equation

$$\frac{\partial^2 \mathcal{P}(x,t)}{\partial x^2} = \gamma\,\ddot{\xi}(x,t) \quad \text{with} \quad \gamma = \frac{\rho\, b_{BM}}{A/2} \tag{7}$$

and with constants ρ denoting the density of the cochlear fluid, b_{BM} the width of the membrane and A the diameter of the cochlea. The initial and boundary conditions for these differential equations are

$$\begin{cases} \xi(x,0) = 0\,, & \dot{\xi}(x,0) = 0 \\ \dfrac{\partial \mathcal{P}}{\partial x}(0,t) = P(t)\,, & \mathcal{P}(1,t) = 0 \end{cases} \tag{8}$$

for a known bounded function $P(t)$ with $P(t) = 0$ for $t < 0$. Together, equations (6) and (7) form a system of coupled ODEs describing how the cochlea responds to sound input. A comparison between the simulations generated using the model and the actual OAEs can be seen in figure 3.

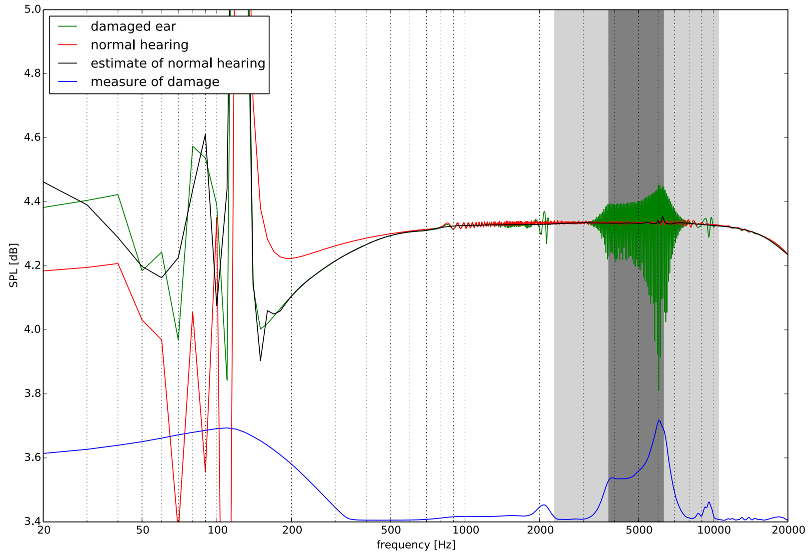

Figure 3: *Comparison of OAEs of normal hearing, damaged cochlea and estimated damage.*

3.1 The Fourier transform of the ODE

The incoming pressure wave is a representation of sound. It is therefore natural to use the frequency domain by means of the Fourier transformation in space. Let us use $\tilde{G}(x, \omega)$ to denote the Fourier transform of a function $G(x, t)$ with angular frequency ω. Then equations (6) and (7) are transformed into the system

$$\begin{cases} \tilde{\mathcal{P}}(x,\omega) = \left[-\omega^2 + i\omega\frac{c_0 c_2}{\tau} + \left(\frac{c_2}{\tau}\right)^2\left(1 + c_1 e^{-i\omega\tau}\right)\right]\tilde{\xi}(x,\omega) \\ \dfrac{\partial^2\tilde{\mathcal{P}}(x,\omega)}{\partial x^2} = -\omega^2\gamma\,\tilde{\xi}(x,\omega). \end{cases} \tag{9}$$

This system can be restated as

$$\begin{cases} \gamma\tilde{\mathcal{P}}(x,\omega) = \left[1 - i\frac{c_0 c_2}{\omega\tau} - \left(\frac{c_2}{\omega\tau}\right)^2\left(1 + c_1 e^{-i\omega\tau}\right)\right]\dfrac{\partial^2\tilde{\mathcal{P}}(x,\omega)}{\partial x^2} \\ \dfrac{\partial^2\tilde{\mathcal{P}}(x,\omega)}{\partial x^2} = -\omega^2\gamma\,\tilde{\xi}(x,\omega). \end{cases} \tag{10}$$

The first identity of this system has sufficient boundary conditions from the Fourier transformed boundary conditions of $\mathcal{P}$ in (8). The second identity is satisfied by the first without imposing the boundary conditions of ξ in (8). Hence the unused boundary conditions must be satisfied automatically from the differential equation for $\tilde{\mathcal{P}}$. We can investigate this by solving the characteristic equation of the differential equation, which is

$$\gamma = \left[1 - i\frac{c_0 c_2}{\omega\tau} - \left(\frac{c_2}{\omega\tau}\right)^2 \left(1 + c_1 e^{-i\omega\tau}\right)\right] \lambda(\omega)^2. \tag{11}$$

The solution for $\tilde{\mathcal{P}}(x, \omega)$ and the boundary conditions $\xi(x, 0)$ and $\dot{\xi}(x, 0)$ are then equal to

$$\tilde{\mathcal{P}}(x, \omega) = -\frac{\tilde{P}(\omega)}{\lambda(\omega)} \frac{\sinh[(1-x)\lambda(\omega)]}{\cosh[\lambda(\omega)]} \tag{12}$$

$$\xi(x, t) = \frac{1}{\gamma\sqrt{2\pi}} \int_{-\infty}^{\infty} \lambda(\omega) \frac{\tilde{P}(\omega)}{\omega^2} \frac{\sinh[(1-x)\lambda(\omega)]}{\cosh[\lambda(\omega)]} e^{i\omega t} d\omega \tag{13}$$

$$\dot{\xi}(x, t) = \frac{i}{\gamma\sqrt{2\pi}} \int_{-\infty}^{\infty} \lambda(\omega) \frac{\tilde{P}(\omega)}{\omega} \frac{\sinh[(1-x)\lambda(\omega)]}{\cosh[\lambda(\omega)]} e^{i\omega t} d\omega \tag{14}$$

These identities imply that the cochlea has no spontaneous excitation modes without a forcing pressure $\tilde{P}(\omega) \neq 0$. Hence the condition $P(t) = 0$ for $t < 0$ guarantees the remaining boundary conditions due to causality, which is the property used in the Zweig paper to justify the delay term in equation (4) from data.

An advantage of the introduction of $\lambda(\omega)$ in equation (11) is the possibility to extend it to the form $\lambda(x, \omega)$ in a more general model where the constants c_0, c_1 or c_2 become functions of x. The function ξ is then still given by equation (13) by substituting $\lambda(x, \omega)$ instead of $\lambda(\omega)$. A second advantage of equation (11) is the existence of real and imaginary parts of λ, which allow not only oscillations, but decay as well. This decay will depend on the frequency, which reflects the frequency-position relationship of the oscillator response in the cochlea.

4 Input Functions

The physical model given in the introduction, derived in the theoretical background and explained in the physical interpretation is that of a resonance cavity, which resembles the cochlea and resonates due to an input pressure function. This input pressure is a representation of a sound wave. The reason INCAS[3] is interested in this model is to improve the current methods of diagnosing hearing loss. In these methods, a certain sound pulse is created and used with a certain procedure to determine the hearing loss. The quality of the diagnosis is therefore highly dependent on the input sound pulse, measurement accuracy and measurement precision. Therefore, we tried to tackle the following problem. Is it possible to improve the input pulse to obtain a result faster without decreasing the accuracy or precision of the diagnosis?

Due to finite time measurements, the input function must be a pulse, which implies that it must be a function with compact support. Furthermore the Fourier transform of the input pulse may not have any zeros. This condition is necessary since hearing loss could occur at any frequency. Furthermore, hearing loss is determined as a spectral response loss, which implies that accuracy requirements need non-zero spectrum. Finally, we can observe only an interval of the frequency domain. It is therefore desirable that the Fourier transform of the input pulse is rapidly decaying in a known way outside the measurable interval of the frequency domain. A second desirable property would be the ability to modify the pulse such that a certain interval in the frequency domain can be examined. Hence, an input pulse must satisfy the following:

1. Compact support in the time domain.
2. Fourier transform without zeros.
3. Rapidly decaying Fourier transform.
4. Adjustable for having values above a given threshold for a given frequency interval.

A simple family of functions $G(t)$ which satisfy the requirements above are sums of a finite, symmetric interval part of the $\operatorname{sech}(t)$. Is is obvious that these functions have compact support. Additionally, their Fourier transform does not have zeroes due to the invariance of the sech under Fourier transform, which smoothes out all the zeroes due to the window. It is easy to see that they are rapidly decaying due to the properties of the sech by using the Riemann-Lebesgue lemma. Finally, we can adjust them at will since the sech is symmetric in the frequency domain and therefore sums of a symmetric finite interval part can cut away small frequency intervals for a given threshold.

Let us define by $\operatorname{rect}(a)(t)$ the unit function on the interval $[-a, a]$ and zero elsewhere. Then the function G and its Fourier transform $\tilde{G}$ are given by

$$\begin{aligned} G(a)(t) &= \sqrt{\frac{\pi}{2}}\operatorname{sech}(t)\operatorname{rect}(a)(t), \\ \tilde{G}(a)(\omega) &= \arctan\left(e^{a/(2\pi)-\omega}\right) - \arctan\left(e^{-a/(2\pi)-\omega}\right). \end{aligned}$$

Using the function G as a building block one obtains a new function S which has the property of selecting an interval in the frequency for which $\tilde{S}$ is a threshold.

$$\begin{aligned} S(a,b)(t) &= G(a)(t) - \frac{1}{|b|}G(a)(t/b), \\ \tilde{S}(a,b)(\omega) &= \tilde{G}(a)(\omega) - \tilde{G}(a)(b\,\omega). \end{aligned}$$

If the threshold is equal to ϵ, then the interval endpoints are the positive

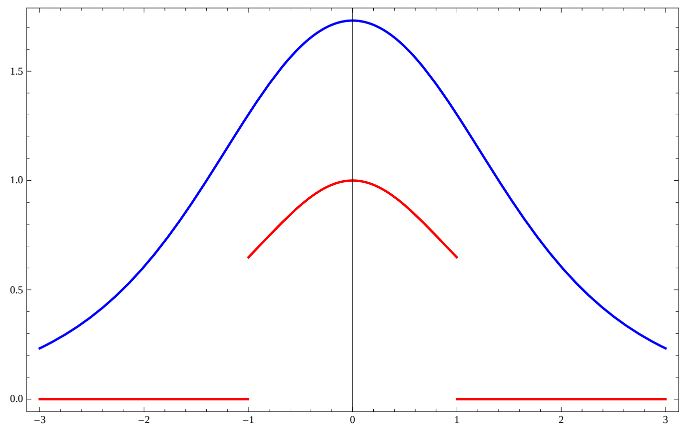

Figure 4: *Red line: plot of sech(t) multiplied with rect$(1)(t)$. Notice the compact support, the discontinuity at ±1 and the absense of roots inside the window. Blue line: Fourier transform of the function above. This function also does not have any zeroes and is rapidly decaying.*

zeros of

$$\tan(\epsilon) = \frac{\sinh(a/\pi)\cosh(\omega) - \sinh(a/\pi)\cosh(b\,\omega)}{\sinh^2(a/\pi) + \cosh(\omega)\cosh(b\,\omega)}.$$

We can always find two positive zeroes for small enough ϵ. Hence S satisfies the properties of a desirable input pulse.

The existence of such a simple function that satisfies the properties needed is very important, as it will allow a relatively simple design and execution of experiments to locate where the damage lies in the frequency spectrum of the ear. Further research would help in refining the definition and use of the input function, thus drastically improving the way hearing damage diagnosis is performed.

5 Parameter estimation

5.1 Toy problem

Determining the value of a set of parameters using some experimental data about the solution of a problem is usually called the *inverse problem.* When experimental data are not available, numerical simulations may also be used. We will describe two different methods that are commonly used to attack inverse problems, namely the finite differences method and the adjoint method. Both share a common foundation, as in both cases we try to minimize an objective function that usually calculates the difference between the current set of parameters and the 'perfect' one. To illustrate these methods let us consider a simple example:

$$D(x)\frac{d^2T(x)}{dx^2} = 1, \quad x \in [0,1] \tag{15}$$

with boundary conditions $T(0) = 1$, $T(1) = 0$.

Suppose we have a given $\tilde{T}$ which satisfies (15), for some unknown $D(x)$. The inverse problem is to calculate the function $D(x)$ that corresponds to $\tilde{T}$. We define an objective function, F, which measures the difference between the exact solution $\tilde{T}$ and our approximation T; minimizing this function is now our goal, and the value of the parameter D corresponding to the minimum of the error function will be the best estimation, at least locally. The inverse problem is therefore tackled using an optimization procedure.

We will make use of gradient-based optimization algorithms, a subset of the class of line search methods, an optimization strategy based on two steps:

1. Find a direction along which F decreases rapidly.
2. Compute a step size which determines how far we should move along that direction.

It is obvious that successful use of a line search method requires the determination of both the direction and the step length.

Both strategies illustrated here, the finite differences method and the adjoint method, assume the gradient direction as the decreasing line, namely

$$D^{(k+1)} = D^{(k)} - \underbrace{\gamma}_{\text{step}} \underbrace{d_D F(T,D)}_{\text{direction}},$$

with d_D denoting the derivative with respect to D. However, these methods differ in the way they calculate the gradient. A first approach is to approximate it by finite differences over D. However, this includes the integration of n differential equations at each step, where n is the dimension of D. A more sophisticated approach is to calculate the gradient using the adjoint method, which is significantly cheaper computationally, since at most two differential equations have to be integrated.

Using the procedure discovered above, starting from an initial guess D_0, we obtain a sequence $F(D_n)$ that satisfies

$$F(D_0) \geq F(D_1) \geq \cdots \geq F(D_k) \geq F(D_{k+1}) \geq \dots$$

and converges to a minimum. A significant disadvantage of this method, as in every hill-climbing method, is the risk of getting stuck in a local minimum. Every minimum found is guaranteed to be a global minimum only if F is convex, which is usually either not true or difficult to prove.

5.2 Finite differences

The idea underlying this approach is the discretization of the problem with respect to the spatial variable. A second order approximation of the derivative of the objective function is calculated and used in the gradient descent method in order to find a local minimum. A *discrete* objective function is defined,

$$F(D) = \sum_{j=1}^{n} (T(x_j; D) - \tilde{T}(x_j))^2$$

where T is a vector containing the evaluation of the approximate solution (corresponding to the approximated parameter D) in each node of the discretisation and $\tilde{T}$ is a vector containing the evaluation of the exact solution in the same points.
At each step we compute the gradient using the formula

$$d_D F^{(k)} = \frac{F(D^{(k)} + \varepsilon) - F(D^{(k)} - \varepsilon)}{2\varepsilon}$$

and we then update our parameter value according to

$$D^{(k+1)} = D^{(k)} - \gamma \cdot d_D F^{(k)}.$$

Since the convergence speed of this method can be very slow for a constant step, it is possible to add an iterative method to better adapt the step length γ. One such possibility is through the *backtracking line search,* which is a good compromise between the two opposite goals of obtaining a step size γ which substantially reduces F and decreasing computational cost. A sample algorithm which geometrically reduces γ is the following:

choose $\gamma_0 > 0$ (generally $= 1$); $\rho, c \in (0,1)$;
set $\gamma = \gamma_0$;
while $F(D^{(k+1)}) \geq F(D^{(k)}) + c \cdot \gamma \cdot d_D F^{(k)}$ **do**
| set $\gamma = \rho\gamma$;
end

5.3 Adjoint method

The adjoint method is a way of significantly decreasing the computational cost of calculating the gradient. An introduction to it will be presented here, more details can be found in the book of Vogel (2002).

Let $T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} T_1 \\ \dot{T}_1 \end{pmatrix}$. Equation (15) can then be written as $\dot{T} = \begin{pmatrix} T_2 \\ \frac{1}{D} \end{pmatrix}$ and the inverse problem can be stated as follows:

$$
\begin{aligned}
\underset{D}{\text{minimize}} \quad & F(T;D) = \int_0^1 f(t,D,x)dx, \text{ where } f(T,D,x) = \int_0^1 (\tilde{T}(x) - T(x))^2 dx \\
\text{subject to} \quad & h(T,\dot{T},D,x) = \dot{T} - \begin{pmatrix} T_2 \\ \frac{1}{D} \end{pmatrix} = 0, \\
& g_1(T(0),D) = T(0) - \begin{pmatrix} 1 \\ T_2(0) \end{pmatrix} = 0, \\
& g_2(T(1),D) = T(1) - \begin{pmatrix} 0 \\ T_2(1) \end{pmatrix} = 0.
\end{aligned}
$$

Here D is a vector of unknown parameters, T is a function of x, $h(T,\dot{T},D,x) = 0$ is an ODE in implicit form and $g_1(T(0),D) = 0$, $g_2(T(1),D) = 0$ are the boundary conditions, which are functions of some of the unknown parameters. Being a gradient-based optimization algorithm, the gradient

$$d_D F(T,D) = \int_0^1 [\partial_T f d_D T + \partial_D f] dx$$

has to be calculated. Unfortunately, it is often expensive to compute $d_D T$. The first step in solving this problem is to introduce the Lagrangian corresponding to the optimization problem defined above,

$$\mathcal{L} = \int_0^1 [f(T,D,x) + \lambda^T h(T,\dot{T},D,x)]dx + \mu_1^T g_1(T(0),D) + \mu_2^T g_2(T(1),D).$$

Here λ is a vector of Lagrange multipliers depending on x, and μ_1 and μ_2 are vectors of multipliers corresponding to the boundary conditions. Since h, g_1, and g_2 are zero everywhere by definition, λ, μ_1 and μ_2 can be chosen freely and we have $d_D\mathcal{L} = d_D F$. The main idea is to choose the values of the multipliers in such a way that the total derivative $d_D\mathcal{L}$ is easy to compute. Thus, the derivative of the Lagrangian is

$$\begin{aligned} d_D\mathcal{L} = \int_0^1 [\partial_T f d_D T + \partial_D f + \lambda^T(\partial_T h d_D T + \partial_T h d_D \dot{T} + \partial_D h]dx \\ +\mu_1^T(\partial_{T(0)} g_1 d_D T(0) + \partial_D g_1) + \mu_2^T(\partial_{T(1)} g_2 d_D T(1) + \partial_D g_2). \end{aligned} \quad (16)$$

The integrand contains the terms $d_D T$ and $d_D \dot{T}$, which are both hard to calculate. For the second term, we apply integration by parts

$$\int_0^1 \lambda^T \partial_{\dot{T}} h d_D \dot{T} dx = \lambda^T \partial_{\dot{T}} h d_D T\big|_0^1 - \int_0^1 \left[\dot{\lambda}^T \partial_{\dot{T}} h + \lambda^T d_x\left(\partial_{\dot{T}} h\right)\right] d_D T dx.$$

Substituting this in (16) we obtain the expression

$$\begin{aligned} d_D\mathcal{L} \quad = \quad & \int_0^1 \left[\partial_T f + \lambda^T\left(\partial_T h - d_x \partial_{\dot{T}} h - \dot{\lambda}^T \partial_{\dot{T}} h\right)\right] d_D T \\ & +\partial_D f + \lambda^T \partial_D h dx \\ & +\mu_1^T\left([\partial_{T(0)} g_1 + \lambda^T \partial_{\dot{T}} h]_0 d_D T(0) + \partial_D g_1\right) \\ & +\mu_2^T\left([\partial_{T(1)} g_2 + \lambda^T \partial_{\dot{T}} h]_1 d_D T(1) + \partial_D g_2\right). \end{aligned}$$

Since we are free to choose the multipliers λ, μ_1 and μ_2, let us take

$$\mu_1^T = \lambda^T \partial_{\dot{T}} h|_0 (\partial_{T(0)} g_1)^{-1}$$

$$\mu_2^T = \lambda^T \partial_{\dot{T}} h|_1 (\partial_{T(1)} g_2)^{-1}.$$

This ensures that the first parts of the last two terms vanish. Furthermore, we choose λ such that

$$\partial_T f + \lambda^T(\partial_T h - d_x \partial_{\dot{T}} h) - \dot{\lambda}^T \partial_{\dot{T}} h = 0, \quad (17)$$

which saves us from having to calculate $d_D T$. With this choice of values for the multipliers we obtain

$$d_D\mathcal{L} = \int_0^1 [\partial_D f + \lambda^T \partial_D h]dx + \mu_1^T \partial_D g_1 + \mu_2^T \partial_D g_2. \quad (18)$$

The first order linear ODE h from the optimisation problem stated above can be rewritten as

$$h(T, \dot{T}, D, x) = \dot{T} - A(D)T - b(D),$$

where $A(D) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $b(D) = \begin{pmatrix} 0 \\ D^{-1} \end{pmatrix}$. It follows easily that $\partial_T h = A(D)$ and $\partial_{\dot{T}} h = I$, and $\partial_D h = \begin{pmatrix} 0 \\ D^{-2} \end{pmatrix}$. Furthermore, $\partial_D f$, $\partial_D g_1$ and $\partial_D g_2$ are zero. Substituting these information into (18) leads to

$$d_D \mathcal{L} = \int_0^1 \lambda^T \partial_D h dx = \int_0^1 \lambda_2(x) D(x)^{-2} dx.$$

and the adjoint equation becomes $\partial_T f - \lambda^T A(D) - \dot{\lambda}^T = 0$.

6 Conclusion

6.1 Summary of results

We have investigated several different ways to attack the problem of modelling hearing damage. They all show promise for further investigation.

Regarding the analysis of the given model, it is obvious that Fourier transforming the differential equations has several advantages. For instance, analysis in the frequency domain is often more natural when working with sound. Furthermore, solving the system is now easier, which can be very helpful for implementing a parameter estimation scheme. It is also worth noticing that parameter estimation may not be needed at all. This is because the motivation behind trying to estimate where the cochlea is damaged is to allow us to alter the sound entering the ear in a controlled way, by means of a hearing aid, so that the ear processes an input that is as close to normal as possible. Therefore, instead of trying to locate where the damage lies, it might be more fruitful to be able to calculate this corrected input signal directly. This may be a significantly hard problem in the time domain, perhaps harder than the parameter estimation itself, but using the frequency domain we showed there are ways to attack it.

The gradient descent method discussed in section 5 is one of the simplest ways of solving an inverse problem. Regardless whether the gradient itself is obtained through finite differences or the adjoint method, the choice of algorithm itself is also very important. Simple methods like gradient descent can have problems such as slow convergence or getting stuck at a local minimum. There exist however more sophisticated methods that may be used to remedy these problems. For example, the conjugate gradient method that keeps track of previous step directions could be tried. Additionally, attention should be paid to quasi-newton methods like BFGS that determine the objective function's Hessian.

6.2 Recommendations for future work

- Investigate whether choosing another function m to fit the experimental data leads to a better model of a damaged cochlea.
- Add the calculation of the OAE to the Fourier-based approach.
- Check whether the input function proposed in section 4 works as expected for real OAE measurements.
- Perform parameter estimation with the current cochlear model using the framework laid out in this paper.

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