

Stress distribution during neck formation: An approximate theory

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Abstract

In this paper we investigate the effects of deformation of a metal specimen, which is either a plate or a cylindrical rod in our case. In particular we study neck formation in tensile loading of a plastic metal. We try to generalize the work of Bridgman, who considered a purely two-dimensional geometry, to an effective theory that takes into account some essential three dimensional characteristics. That extending the description of neck formation to three dimensions is necessary was illustrated by recent experimental findings of [1].

We have studied existing models from the literature that describe necking for plates and cylinders to identify the consequences of the crucial assumption of uniform in-plane stress. We also developed a new model that we have not yet been able to analyze. Finally, using work of [4] in which a power law relation between the von Mises stress and the effective strain is used, a perturbation analysis for a simple flat geometry was performed. The perturbation analysis offers a good starting point for generalizing the work of Bridgman to three dimensions.

KEYWORDS: Neck formation, von Mises stress, tensile pulling, plane stress assumption

1 Introduction

In many daily life situations materials are deformed. If deformations are very small the material will respond elastically. However, for metals deforming in collisions the plasticity regime is entered for relatively small deformations and the material will therefore not return to its initial state. This phenomenon can also be observed in uniaxial tension experiments of a specimen, which is in our case a metal bar or cylinder. If in an experiment the length of the bar is continuously increased by exerting a pulling force sufficiently large to accomplish elongation of the metal, then for certain

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loading the metal will yield and enter the plasticity regime. Still further enlarging the load gives rise to so-called "necking". A picture of typical neck occurring in a cylindrical specimen is depicted in Fig. 1.

The stress distributed in the metal has been shown in [1] to become fully three-dimensional, that is, the two-dimensional models originally proposed by Bridgman in the nineteenfifties will not be appropriate to model necking. The goal of this study during the SWI 2013 is to extend the 2D description of Bridgman, so that the findings in [1] are effectively incorporated. Such a description would be very useful in finite element codes for collisions of ships, as full 3D models are computationally very expensive and so an approximate incorporation of the stresses in 2D could drastically improve the computation time needed to analyze such situations. To find an extension of the existing models a good review of the literature and the most important concepts in continuum mechanics were required.

The paper is organized as follows. We first introduce some concepts from continuum mechanics needed for the description of the problem. Next we describe the model that we employed for both a cylindrical and a plane geometry. In section 3, we discuss our preliminary findings. Finally, in section 4, we summarize our results and make recommendations for future research.



Figure 1: A neck appears after applying a critical load to the cylinder.

2 Model

In this section we discuss three different models that were investigated. The first model was constructed from some special assumptions using the general theory and a modelling assumption about the strain rate. We discuss two possible choices for the strain rate. The first possibility was to assume a constant strain rate, the other was found in a paper [4]. The other two models we studied were both taken from

the literature: the original model of Bridgman in [2] and a more recent version by Kaplan in [5]. Before discussing the models we start with reviewing some concepts of continuum mechanics.

2.1 Concepts from continuum mechanics

To understand the problem of necking, we need some concepts of continuum mechanics that we here present. If a material is deformed a *displacement field* results, which is denoted as $\mathbf{u}(\mathbf{x})$. The *strain* ϵ is defined as

$$\epsilon = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T). \tag{1}$$

It is a symmetric tensor that is related to the stress tensor $\sigma(\mathbf{x})$, which assigns a value of the force per unit area to each point \mathbf{x} in the material, by a constitutive relation. In the elastic case the constitutive relation between σ and ϵ is linear. In the regime where the material yields and the deformation is plastic, the situation is much more difficult. However, for the one-dimensional case an empirical relation between stress and strain still exists as we will see.

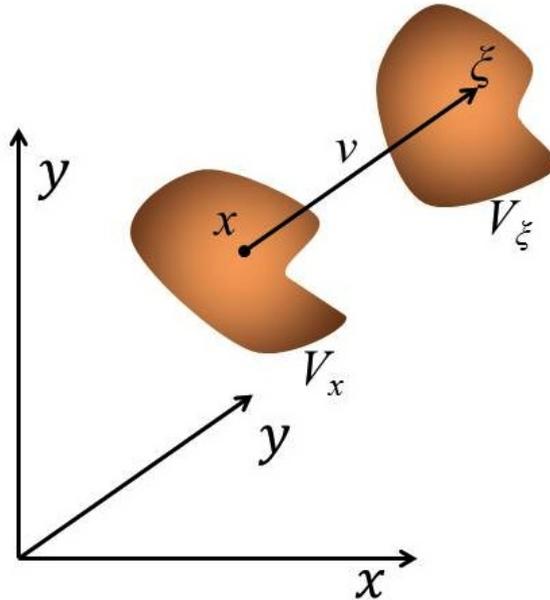


Figure 2: A deformation of a material may lead to a change in volume and stresses throughout the solid.

If we assume that the yielding is unaffected by moderate hydrostatic pressure or

tension, which is correct to a first approximation it follows that the yielding condition only depends on the principal components, or eigenvalues, of the deviatoric stress tensor, σ' , defined by

$$\sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\text{Tr}(\sigma)\delta_{ij}. \quad (2)$$

The eigenvalues of σ' , $\{\sigma'_1, \sigma'_2, \sigma'_3\}$ are not independent since they satisfy

$$\sigma'_1 + \sigma'_2 + \sigma'_3 = 0,$$

as follows immediately from the definition of deviatoric stress. If we further assume that the material is isotropic, the condition for which yielding will occur only depends on the eigenvalues, of which only two are independent. So we can write the equation for yielding

$$F(\sigma'_1, \sigma'_2) = 0, \quad (3)$$

with F an arbitrary function.

Finally, we use the von Mises proposal (1937) which has been verified in a number of experiments that the yielding condition depends quadratically on $\sigma'_1, \sigma'_2, \sigma'_3$. Using symmetry this gives

$$\sigma'^2_1 + \sigma'^2_2 + \sigma'^2_3 = \frac{2}{3}\bar{\sigma}^2, \quad (4)$$

where $\bar{\sigma}$ is called the *von Mises* stress. This nonlinear relation can be expressed in terms of the eigenvalues of the original stress tensor σ as

$$\bar{\sigma} = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}}. \quad (5)$$

It turns out that in the plastic regime the von Mises stress is related to the strain by a power law. For the one-dimensional case this relation is

$$\bar{\sigma} = C\epsilon^N, \quad (6)$$

where C is a material constant and N is a power law exponent whose value is in the range $[0.1, 0.2]$. To generalize the constitutive relation (6) to three dimensions different approaches are possible, that we will discuss in Model I. The original Bridgman model and related to it the model by Kaplan, will be explained in the subsections Model II and Model III.

2.2 Model I

1. Steady motion

In this approach we try to generalize (6) to 3D by defining an effective scalar strain, $\bar{\epsilon}$ such that (6) remains valid when ϵ is replaced by $\bar{\epsilon}$. The problem is that

we need to know the effective strain, which in general depends on the loading history. Even though we do not know how the strain evolves, the strain rates $\dot{\epsilon}$ obey [3]

$$\dot{\epsilon} = \sqrt{\frac{2}{9} ((\dot{\epsilon}_1 - \dot{\epsilon}_2)^2 + (\dot{\epsilon}_1 - \dot{\epsilon}_3)^2 + (\dot{\epsilon}_3 - \dot{\epsilon}_2)^2)}, \tag{7}$$

where the dot denotes differentiation with respect to time. Assuming constant time derivatives, Eq. (7) is also valid for the strains, which implies that the dots in (7) can simply be left out.

In order to close the equations we need two more equations. To this end we invoke the Levy-Mises flow rules which state

$$\frac{\epsilon_1 - \epsilon_2}{\sigma_1 - \sigma_2} = \frac{\epsilon_1 - \epsilon_3}{\sigma_1 - \sigma_3} = \frac{\epsilon_3 - \epsilon_2}{\sigma_3 - \sigma_2}. \tag{8}$$

and give the two necessary conditions to close the system.

2. *Hutchinson theory*

In a paper by Hutchinson et al. [4] it was proposed to generalize relation (6) in the following natural fashion

$$\dot{\epsilon}_{ij} = \frac{3}{2} \alpha \bar{\sigma}^{n-1} \sigma'_{ij}, \tag{9}$$

with α a material constant and n the strain hardening exponent that is typically larger than 1 and directly related to N . We remark that relation (9) is similar to (6) if we require in addition that the the strain rate is time independent and that the strain and the stress have a common set of eigenvectors.

We next discuss two existing models in the literature. One is the original model of Bridgman for necking. The other model is a model introduced by [5].

2.3 Model II: The Bridgman model for necking

Bridgman discusses neck formation in a cylindrical tensile specimen. The distribution of stress across a transverse section is, however, not necessarily uniform. Measurements generally only provide data about the mean stress through the neck. Since the shape of the neck is not known beforehand calculating the stress distribution is extremely difficult and determining its shape from first principles requires tracing the time evolution of the dynamical process of neck formation as is done by [4]. Bridgman made the assumption, based on his own experimental data, that at the neck minimum the stress is uniformly distributed. From the area reduction at the position of the neck, which we call $x = 0$, the strain is known. Furthermore the strain rate can be shown to be proportional to the radial distance r . For equation for equilibrium is again given by

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} = 0, \text{ at } z = 0. \tag{10}$$

The yield condition is also very much simplified in this case, because $\sigma_{rr} = \sigma_{\theta\theta}$ as $\dot{\epsilon}_{rr} = \dot{\epsilon}_{\theta\theta}$ and the strain rate is proportional to r . This gives the yield condition

$$\sigma_{zz} - \sigma_{rr} = Y. \quad (11)$$

Bridgman then lets his principal stress direction in a meridian plane to the axis as in Fig. 2.1. We have

$$\sigma_{zz} \simeq \sigma_3, \quad \sigma_{rr} \simeq \sigma_1, \quad \sigma_{rz} \simeq (\sigma_3 - \sigma_1)\psi. \quad (12)$$

This implies that the yield condition is

$$\sigma_3 - \sigma_1 = Y + \mathcal{O}(\psi), \quad (13)$$

from which it immediately follows that

$$\left(\frac{\partial \sigma_{rz}}{\partial z}\right)_{z=0} = Y \left(\frac{\partial \psi}{\partial z}\right)_{z=0} = \frac{Y}{\rho}. \quad (14)$$

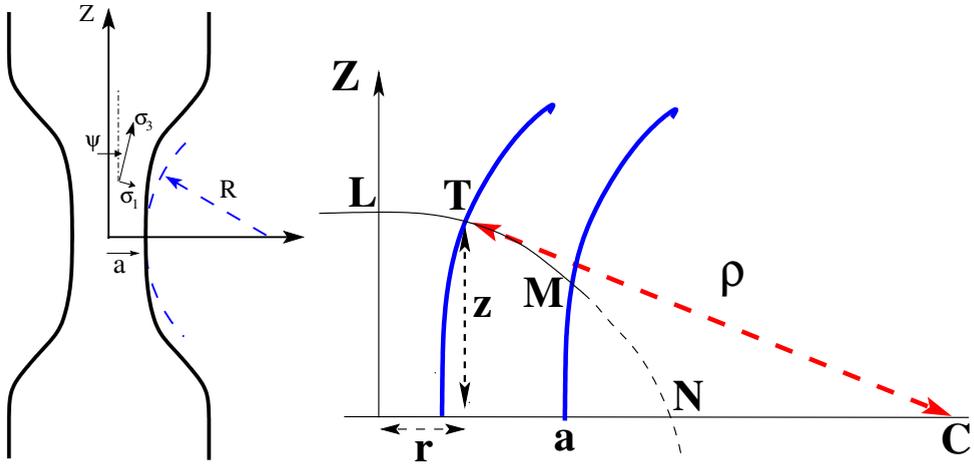


Figure 3: The geometry for the Bridgman model. Neck formation in a cylindrical geometry. Plane and uniform stress assumption at $z = 0$.

This leads to the following partial differential equation for σ_{zz}

$$\frac{\partial \sigma_{zz}}{\partial r} + \frac{Y}{\rho} = 0, \quad \text{for } z = 0. \quad (15)$$

Using the symmetry in Fig. 3, we can deduce that

$$\rho^2 \simeq CT^2 = OC^2 - ON^2 \simeq (r + \rho)^2 - ON^2, \quad (16)$$

hence for any point on OA we have

$$\rho = \frac{a^2 + 2aR - r^2}{2r}. \tag{17}$$

From (17) [2] obtained the formula

$$\frac{\sigma_{zz}}{Y} = 1 + \ln \left(\frac{a^2 + 2aR - R^2}{2aR} \right). \tag{18}$$

2.4 Model III: The Kaplan model

In 1973, M.A. Kaplan [5], extended the analysis of Bridgman. His analysis deals with necking in bars of mild steel. One assumption that is made here is that the displacement will be axially symmetric in a symmetric bar. This reduces the number of unknowns in this analysis drastically. Another important assumption is that the deformation will be produced entirely by plastic flow, so the elastic deformation is neglected. This can be supported by the fact that the elastic contribution to the total axial strain in ductile metals is of the order of 1 percent at the onset of necking, and decreases as necking proceeds.

Kaplan uses the work of Bridgman to calibrate his own model. Experimentally, Bridgman concluded that during necking, the ratio between the external radius and core radius remains constant during necking from the point on where necking has started. This means that $r/a = r_0/a_0$ where r and r_0 are the deformed and initial radial positions of a particle respectively. The radius of the profile, a is measured on a plane on which the particle lies after deformation. The initial radius of the bar is a_0 . This fixed quantity implies an equation for the radial displacement in terms of the profile of the bar a , which is assumed to be a function of the z -direction (see Fig. 1) and time t . Kaplan then performs a formal analysis to derive the strain fields and strain rates in terms of this profile.

From the strain field, the Levy-Mises plasticity equations and use of the radial displacement uniformity, the stress field could be determined. This is a closed system of eight equations and unknowns. After some formal derivations the predicted profile of the neck, $a(z, t)$ is determined and has the shape of a parabola.

The analysis of [5] is valid throughout a significantly larger portion of the plastic flow region than the analysis of [2]. The analysis holds only for a necked cylindrical mild steel tensile specimen. However, experiments show a high similarity for ductile metals and Kaplan argues that his results apply to those materials as well.

A nice feature of Kaplan's work is that all parameters used in his model can be measured very well. Also, his results have good agreement with experiments and tensile tests.

3 Results

To understand the necking process in time we studied a thin sheet as sketched in Fig. 4. In this study we use the formalism as presented in [4]. The sheet is initially in

rest and then a force P per unit length is applied at the ends, which are initially at $x = -L$ and $x = L$. For simplicity we assume $L \gg 1$. The width of the sheet is given by $h(x, t)$. The stress is linearly related to P by at each section $x_1 = \text{constant}$

$$P = \sigma_{11}h \quad (19)$$

We next make the plane stress assumption, that is, all quantities are independent of x_2 . Moreover only the σ_{22} and σ_{11} do not vanish and these are taken to be *uniform* over a section with constant x_1 . Finally, we assumed symmetry with respect to the x_3 coordinate and imposed volume conservation, that is,

$$\dot{\epsilon}_{11} + \dot{\epsilon}_{22} + \dot{\epsilon}_{33} = 0, \quad (20)$$

with the additional condition that $\dot{\epsilon}_{22} = 0$, as there is no x_2 dependence. If we use Eq. (9), we find after calculating the von Mises stress

$$\dot{\epsilon}_{11} = -\dot{\epsilon}_{33} = \frac{\alpha\sqrt{3}}{2} \left(\frac{\sqrt{3}P}{2h} \right)^n. \quad (21)$$

We can now find the evolution equation of $h(x_1, t)$, by calculating the time derivative of h with respect to time, keeping in mind that there will also be a convective contribution, that is

$$\dot{h} = \frac{\partial h}{\partial t} + v_1 \frac{\partial h}{\partial x_1} = \dot{\epsilon}_{33}h. \quad (22)$$

In Eq. (22) we introduced $v_1(x_1)$, which is the velocity in the x_1 -direction. If we now use Eq. (21), we have derived an evolution equation for $h(x_1, t)$, which however includes the velocity v_1 .

3.1 Perturbation analysis

To find an approximate solution to Eq. (22) we perform a perturbation analysis. We will repeat here the reasoning of [4]. First we assume that the sheet is perfect and therefore $h(x_1, t)$ only depends on t , next we add a small sinusoidal perturbation, so we can write

$$h(x_1, t) = h_0(t) \left(1 - \xi \cos \left(\frac{2\pi x_1}{l} \right) \right), \quad (23)$$

where ξ is a small parameter. The solution to zero order in ξ satisfies

$$\dot{h}_0 = -\frac{1}{2}\sqrt{3}\alpha \left(\frac{\sqrt{3}P}{2} \right)^n h_0^{1-n} = -h_0 f(h_0), \quad (24)$$

where we introduced $f(h_0) = \dot{\epsilon}_{11}$ for notational convenience. Eq. (24) is easily solved as

$$h_0(t) = \left[h_0^n(0) - \frac{n}{2} \sqrt{3} \alpha \left(\frac{\sqrt{3} P}{2} \right)^n t \right]^{\frac{1}{n}}. \tag{25}$$

The first order contribution in ξ can be obtained by substituting Eq. (21) in Eq. (22) and next differentiating with respect to x_1 . This yields

$$\frac{\partial^2 h}{\partial x_1 \partial t} \frac{\partial h}{\partial x_1} - \frac{\partial^2 h}{\partial x_1^2} \frac{\partial h}{\partial t} = \left(\frac{\partial h}{\partial x_1} \right)^2 [-f'(h) - f(h)] + f(h) \frac{\partial^2 h}{\partial x_1^2}, \tag{26}$$

with f as defined in (24). We could try to solve the nonlinear equation (26) numerically, but we have to keep in mind that that this equation is only valid for in-plane stress and the strain rates only depend on x and not on z . If we do make such an assumption then solving Eq. (26) would determine how a perturbation $h(x, t)$ would evolve in time. For reasons of time we have not numerically solved (26), but rather delved deeper in the theory behind neck formation closely following [4].

Of course, like in [4] it is possible to substitute the sinusoidal expression for h (23) and see keeping only terms linear in ξ to calculate the linear variation of the $h(x_1, t)$ in time as a consequence of the convective term. We will not repeat this calculation here, but rather try to determine the functional form of h when a perturbation is introduced.

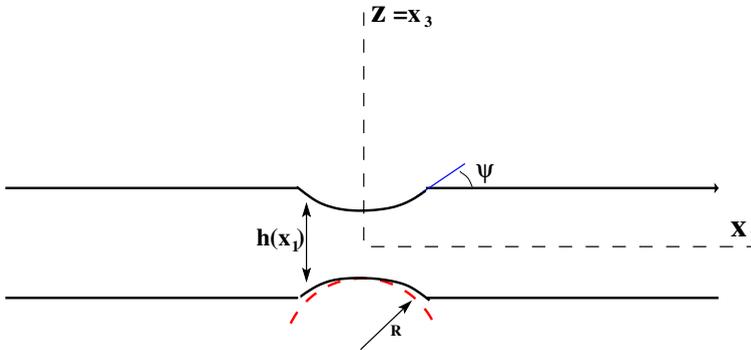


Figure 4: A perturbation analysis can help to calculate the initial neck shape, without assumptions on the curvature.

Assume now that the wavelength of the perturbation is very large and define $X = \beta x_1$, where β is of the order of the inverse wavelength as in [4]. We use X and $z = x_3$ as coordinates. In the creeping flow approximation we can use $\text{div } \sigma = 0$, as

an equilibrium condition. In components this reads

$$\beta \frac{\partial \sigma_{11}}{\partial X} + \frac{\partial \sigma_{13}}{\partial z} = 0 \quad (27)$$

$$\beta \frac{\partial \sigma_{13}}{\partial X} + \frac{\partial \sigma_{33}}{\partial z} = 0 \quad (28)$$

The boundary condition at $z = h(X)/2$ is given by

$$\begin{aligned} -\sigma_{11} \sin \psi + \sigma_{13} \cos \psi &= 0 \\ -\sigma_{13} \sin \psi + \sigma_{33} \cos \psi &= 0 \end{aligned} \quad (29)$$

where $\tan \psi = \beta h'(X)/2$ and the prime denotes differentiation with respect to X . The boundary conditions can be expanded up to order β^2 as well as the stresses

$$\begin{aligned} \sigma_{11} &= \sigma^{(0)}(X) + \beta \sigma_{11}^{(1)} + \beta^2 \sigma_{11}^{(2)} + \dots \\ \sigma_{33} &= \beta \sigma_{33}^{(1)} + \beta^2 \sigma_{33}^{(2)} + \dots \\ \sigma_{13} &= \beta \sigma_{13}^{(1)} + \beta^2 \sigma_{13}^{(2)} + \dots \end{aligned} \quad (30)$$

and the strain rates

$$\begin{aligned} \dot{\epsilon}_{11} &= -\dot{\epsilon}_{33} = \dot{\epsilon}(X) + \beta \dot{\epsilon}_{11}^{(1)} + \beta^2 \dot{\epsilon}_{11}^{(2)} + \dots \\ \dot{\epsilon}_{13} &= \dot{\epsilon}_{13}^{(0)} + \beta \dot{\epsilon}_{13}^{(1)} + \beta^2 \dot{\epsilon}_{13}^{(2)} + \dots, \end{aligned} \quad (31)$$

where $\epsilon_{13}^{(0)} = 0$, but is kept for clarity as in [4].

The strain rates are related to the flow velocity in the following way

$$\dot{\epsilon}_{11} = \beta \frac{\partial v_1}{\partial X} \quad \dot{\epsilon}_{33} = \frac{\partial v_3}{\partial z} \quad 2\dot{\epsilon}_{13} = \frac{\partial v_1}{\partial z} + \beta \frac{\partial v_3}{\partial X}. \quad (32)$$

To go beyond the in-plane plane stress assumption we would need to take into account $\dot{\epsilon}_{22}$, which could be achieved in a perturbative approach. Of course, this would make the equations much more difficult to solve, but in this way a good estimate of non in-plane effects can be given.

We next continue with the Hutchinson analysis. A major simplification of $\dot{\epsilon}_{22} = 0$ is that we can express σ_{22} in terms of σ_{11} and σ_{33} as

$$\sigma_{22} = \frac{\sigma_{11} + \sigma_{33}}{2}.$$

If we write all expressions up to order β^2 , we find the following values of the stress and strain rates

$$\begin{aligned} \sigma_{11} &= \sigma^{(0)} \left[1 + \frac{\beta^2 (n-2) h h''}{12n} \left(1 - 12 \frac{z^2}{h^2} \right) \right] \\ \sigma_{33} &= \sigma^{(0)} \frac{\beta^2 h h''}{8} \left(1 - 4 \frac{z^2}{h^2} \right) \\ \dot{\epsilon}_{11} &= \dot{\epsilon}^{(0)} \left(1 - \frac{\beta^2 h h''}{24} \right) \left(n + 4 + 12(n-4) \frac{z^2}{h^2} \right) \end{aligned} \quad (33)$$

From Eqs. (33) it can be seen that only when the strain hardening exponent n equals 4, $\dot{\epsilon}_{11}$ will be uniform across the neck.

Furthermore, we can now compare the result in [4] with that in [2] by introducing the radius of curvature as

$$\frac{1}{R} = \frac{\beta^2}{2} h'' . \tag{34}$$

By eliminating h from the expression for σ_{11} and σ_{33} we obtain

$$\begin{aligned} \frac{\sqrt{3}\sigma_{11}}{2\bar{\sigma}} &= 1 + \frac{h}{4R} \left[1 - \frac{z^2}{h^2} \right] \\ \frac{\sqrt{3}\sigma_{33}}{2\bar{\sigma}} &= \frac{h}{4R} \left[1 - \frac{z^2}{h^2} \right] , \end{aligned} \tag{35}$$

which agree exactly with the Bridgman expressions to order $\frac{z^2}{h^2}$.

4 Conclusions and recommendations

We conclude that the problem of neck formation is far from trivial. In order to find a generalization of the Bridgman result we studied 3 different models. The model of Kaplan is interesting and may prove very useful, however, we have not been able to generalize this to more dimensions. We constructed a model using the assumption of constant strain rates, which makes it much easier to take the convective terms into account. Finally, we found a study of [4], which appears to be a good alternative to the Bridgman theory. This model takes into account time dependent strain rates and can indeed be generalized to cases in which there is no assumption made about the stresses all being in-plane. Unfortunately, time has not permitted to do the complete analysis, but a perturbation analysis along similar lines as that in [4] would open new avenues for the resolution of the problem of taking the necking problem to three dimensions.

Another direction which may be fruitful, is to start with one of the constitutive models proposed in this study and investigate them numerically. Comparison between experimental data and modeling results would indicate which constitutive relation would be best. Next, complementary to the perturbation analysis, a numerical study of the nonlinear model could be performed at reasonable computational costs, so that a good estimate of the errors resulting from the assumptions such as a uniform stress distribution across the neck can be obtained.

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