

# The random disc thrower problem

Ted van der Aalst      Dee Denteneer      Hanna Döring  
Manh Hong Duong      Ross J. Kang\*      Mike Keane      Janne Kool  
Ivan Kryven      Thomas Meyfroyt      Tobias Müller      Guus Regts  
Jakub Tomczyk

September 3, 2014

## Abstract

We describe a number of approaches to a question posed by Philips Research, described as the “random disc thrower” problem. Given a square grid of points in the plane, we cover the points by equal-sized planar discs according to the following random process. At each step, a random point of the grid is chosen from the set of uncovered points as the centre of a new disc. This is an abstract model of spatial reuse in wireless networks. A question of Philips Research asks what, as a function of the grid length, is the expected number of discs chosen before the process can no longer continue?

Our main results concern the one-dimensional variant of this problem, which can be solved reasonably well, though we also provide a number of approaches towards an approximate solution of the original two-dimensional problem. The two-dimensional problem is related to an old, unresolved conjecture ([6]) that has been the object of close study in both probability theory and statistical physics.

KEYWORDS: generating functions, Markov random fields, random sequential adsorption, Rényi’s parking problem, wireless networks

## 1 Introduction

Various algorithms have been developed for effectively and efficiently maintaining and updating network information in wireless sensor networks. In order to analyse the performance of certain stochastic gossiping algorithms, Philips Research has posed the following two questions.

The first question is related to the maintenance of network information in wireless sensor networks. Consider an  $n \times n$  grid of evenly spaced points (with spacing 1). A disc thrower sequentially distributes (closed) discs of fixed radius  $r$ , so that each

---

\*Corresponding author.

disc is centred on a grid point randomly and uniformly chosen from grid points that were not covered by one of the previous discs. (Discs are allowed to overlap.) The disc thrower continues throwing discs until every grid point is covered by at least one disc. The question is, how many discs is the disc thrower expected to throw? More specifically, what is the probability distribution for the number of discs thrown?

A second question is related to the propagation of new network information in wireless sensor networks. Again consider an  $n \times n$  grid. The disc thrower now throws discs in phases. Again he throws discs sequentially. During each phase though he is only allowed to throw discs so that each disc is centred on a grid point that he covered during the previous phase. Once he cannot throw any more discs, he starts his next phase. During the first phase he is only allowed to throw a disc on one of the corner points. The question of interest here is, what are the expected number of phases until the disc thrower covers the entire grid?

We will primarily focus on the first problem. As such, let us precisely define the parameters that we are interested here. Let us first note that there is a natural  $d$ -dimensional generalisation of the (first) random disc thrower problem. Consider a  $d$ -dimensional  $n \times \dots \times n$  grid with  $n^d$  equally spaced points with spacing 1. A sphere thrower sequentially distributes (closed)  $d$ -dimensional spheres of radius  $r$  such that each sphere is centred on a grid point chosen uniformly at random from grid points not contained within a previously thrown sphere. Let  $N_d(n, r)$  be the number of spheres thrown by the end of this process. We are most interested in the *coverage ratio*

$$\theta_d(n, r) := \frac{\mathbb{E}[N_d(n, r)]}{n^d}.$$

Since our first interest is in the case  $d = 2$  that corresponds to discs, we often drop the subscript and write  $N(n, r)$  ( $= N_2(n, r)$ ) and  $\theta(n, r)$  ( $= \theta_2(n, r)$ ).

The structure of the report is as follows. We start by discussing some of the related background literature, then present some simulation results that give us insights on how the expected number of thrown discs is related to the grid size  $n$  and the discs' radius  $r$ . This is followed by an analysis of the one-dimensional disc thrower problem, where exact results can be obtained. We also propose two approximation methods based on Markov chains and enumeration of all possible fillings of the grid and discuss them for the one-dimensional case. We then discuss the possibility of extending these methods to the two-dimensional grid, in the specific case of discs having radius  $r = \sqrt{2}$ . We leave the extension to larger discs for later work. At the end, we briefly look at the second problem and discuss how it can be solved in the one-dimensional case.

## 1.1 Random sequential adsorption and Rényi's parking problem

The (first) disc thrower problem is closely related to some models of physical chemistry. Perhaps the most relevant is that of random sequential adsorption (RSA) which we describe now. Suppose we have some surface and a sequence of particles land at random locations on the surface. Each particle adheres to the surface, or is "adsorbed",

if it lands at an exposed portion of the surface. In particular, each adsorbed particle covers a region of the surface, which prevents the adsorption of any particle that lands there afterwards. The process is irreversible, meaning that the system will eventually arrive at a *jammed state*, after which no new particles may be adsorbed. The standard models that incorporate these kinds of stochastic and geometric elements are known as *random sequential adsorption (RSA)*. For a comprehensive overview of RSA, the reader is referred to [4]. Most of the research done in this field is about determining the *coverage ratio*  $\theta$  in the eventual jammed state: for adsorption onto lattices,  $\theta$  is the ratio of adsorbed particles to the number of lattice points; in continuous space,  $\theta$  is the density of points of adsorption in the surface. Clearly, the random disc thrower problem is captured by an appropriate lattice RSA model.

Although Philips is mainly interested in the lattice version of the problem, we mention here that the continuous RSA may be viewed as a useful limiting case. In lattice RSA, discs of small radius correspond to regions of the lattice that are jagged and far from round. On the other hand, taking  $r$  large and  $n$  even larger, the shapes more closely approach perfectly circular shapes in a continuous square. In particular, the random disc thrower problem can in this case be approximated by the continuous disc thrower problem, in which perfectly circular (closed) discs of radius  $r$  are randomly and sequentially thrown into the plane, so that their centres land inside a square of side length  $x$ , but do not land on a previously covered point, continuing until no area of the square is uncovered. Let us denote the analogously defined coverage ratio by  $\tilde{\theta}(x, r)$  and write  $\tilde{\theta}(r) = \lim_{x \rightarrow \infty} \tilde{\theta}(x, r)$ . This problem is also known in the literature as continuous RSA with hard discs of radius  $r/2$ . Figure 1 depicts a jammed state in this model.

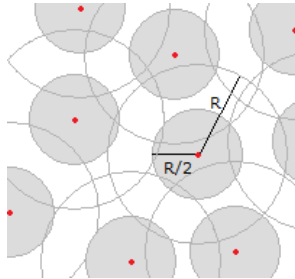


Figure 1: RSA with hard discs of radius  $r/2$  in jammed state.

This leads us to another very relevant but older model, one that is related to the random car parking problem solved by [9]. Consider the  $d$ -dimensional cube  $[0, x]^d$  for some  $x \geq 1$ . First we place a random unit-length  $d$ -dimensional cube so its centre point is uniformly distributed in  $[1/2, x - 1/2]^d$ . Subsequently, we place a new  $d$ -dimensional cube randomly and independently within  $[0, x]^d$  in such a way that it does not intersect any previously placed  $d$ -dimensional cube. We repeat this until no more cubes can be placed. The question is, what is the expected eventual

density of cubes placed in  $[0, x]^d$ ? This model may be referred to as the (continuous) *random sequential packing* of  $d$ -dimensional cubes, and such problems are discussed more generally in [3].

The random sequential packing problem for  $d$ -dimensional cubes is quite natural and is related to a wide variety of problems in statistical physics. By a suitable simple rescaling and ignoring boundary effects in the asymptotic limit, one may consider the  $d = 2$  case as equivalent to a continuous and “square” version of our random disc thrower problem. Since the objects are simpler, one might expect that this problem is somewhat easier than the disc thrower problem; however, the process retains the same irreversibility property, which seems to be a fundamental difficulty. [9] solved the  $d = 1$  case, known best as the random car parking problem, by proving that the expected density in the limit as  $x \rightarrow \infty$  has the form

$$C_1 := \int_0^\infty \exp\left(-2 \int_0^t \frac{1 - e^{-u}}{u} du\right) dt \approx 0.748.$$

The evocative name of this model comes from imagining the intervals as cars of unit length that are parked randomly in a street of length  $x$ . [6] considered the  $d$ -dimensional problem in general and conjectured that the density should converge to  $C_1^d$ . From simulation results, this conjecture is widely believed to be false for every  $d > 1$ . Nonetheless, it remains open after more than half a century! Even the *existence* of the  $d$ -dimensional limit density was not proven until [7].

Following on that work, [8] proved a law of large numbers and central limit theorem for a general class of lattice RSA models, of which the random disc thrower problem is a member. Those results imply the following.

**Theorem 1.1** ([8]). *For all  $r \in \mathbb{R}^+$  there is a constant  $\theta = \theta(r) \geq 0$  such that for all  $p \in [1, \infty)$*

$$\frac{N(n, r)}{n^p} \xrightarrow{L^p} \theta \text{ as } n \rightarrow \infty.$$

**Theorem 1.2** ([8]). *For all  $r \in \mathbb{R}^+$  there is a constant  $\sigma = \sigma(r) \geq 0$  such that*

$$\frac{\text{Var}[N(n, r)]}{n^2} \rightarrow \sigma \text{ as } n \rightarrow \infty$$

and

$$\frac{N(n, r) - \mathbb{E}[N(n, r)]}{n} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma) \text{ as } n \rightarrow \infty.$$

Hence for  $n$  large enough the distribution of  $N(n, r)$  is approximately Gaussian. Moreover, the coefficient of variation tends to zero as  $n^{-1}$ . Thus a good approximation for grids of moderate size is  $\mathbb{E}[N(n, r)] \approx n^2\theta(r)$ .

## 1.2 Simulation estimates of $\theta(r)$

In this subsection, we discuss heuristic and simulation estimates for the coverage ratio  $\theta(r)$  in terms of  $r$ . Let us consider first the case  $r = 1$ . A disc thrown on a grid point

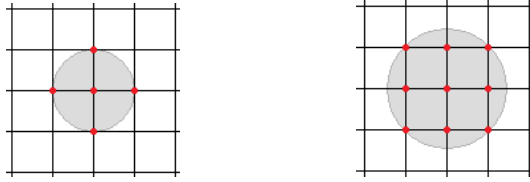


Figure 2: A disc with radius  $r = 1$  or with radius  $r = \sqrt{2}$ .

will then cover four neighbouring nodes, see Figure 2, left. Hence, in this case, the problem is equivalent to randomly selecting grid points in sequence, subject to the restriction that each selected point does not neighbour a previously selected point. This problem is known as RSA of monomers on a square lattice with nearest-neighbour exclusion. An approximation for  $\theta(1)$  derived from series analysis has been obtained (cf. [10]) with the estimate  $\theta(1) \approx 0.36413$ . Ignoring effects of the grid boundary, or taking  $n$  sufficiently large, we can then estimate  $\mathbb{E}[N(n, 1)] \approx 0.36413n^2$ . Next let us consider the case  $r = \sqrt{2}$ . A disc thrown on a grid point will then cover eight nearby nodes, see Figure 2, right. This problem is known as RSA of monomers on a square lattice with next-nearest-neighbour exclusion. One can deduce that this problem corresponds to RSA of hard unit squares on a square lattice ([4], page 1310). The coverage for the latter problem is estimated as  $\theta \approx 0.7476$ . However, since a square covers four points and a square corresponds to one disc, we find for our problem  $\theta(\sqrt{2}) \approx 0.7476/4 = 0.1869$ , giving the estimate  $\mathbb{E}[N(n, \sqrt{2})] \approx 0.1869n^2$ .

We are not aware of published analysis to approximate the coverage ratio for other disc sizes. Simulations could help to determine the coverage. Moreover, methods using rate equations to estimate the coverage are discussed in [4] and [10].

Using continuous RSA as an approximation however, we can develop asymptotic estimates of  $\theta(r)$  for large  $r$ . The continuous RSA problem has been studied extensively and the coverage has been estimated by  $\hat{\theta}(1/\sqrt{\pi}) \approx 0.5479$  (cf. [10]). Since each disc has area  $\pi r^2/4$ , we deduce that  $\theta(r) \approx 0.5479/(\pi r^2/4) \approx 0.6976r^{-2}$  as  $r$  grows large. This approximation and some simulation results are plotted in Figure 3 for  $n = 128$  against  $r$ , where for each  $r$  we average over 100 simulations. The approximation becomes more accurate for larger  $r$  as the discrete discs become more circular.

We end this section with a short remark about the effect of the grid boundary on the covering ratio, i.e. the difference between  $\theta_2(n, r)$  and  $\theta_2(r)$ . (The boundary effect is also the essential difference between RSA and random sequential packing.) Points near the boundary are more likely to be covered than points in the interior of the grid. However, in [4] it is noted that in the one-dimensional case these boundary effects decay superexponentially. Furthermore, simulations suggest that for the two-dimensional case boundary effects tend to decay even faster than in the one-dimensional case. Thus, in moderate size grids, the boundary has only a small influence on the actual coverage ratio, when compared with the coverage ratio for

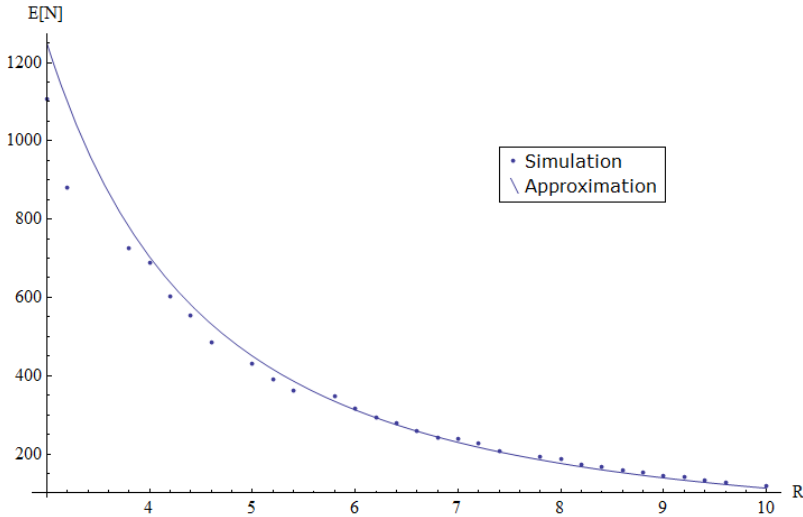


Figure 3: Asymptotic behaviour of  $\mathbb{E}[N]$ .

large  $n$ . In Figure 4, simulation results are shown for the coverage ratio with respect to the position of grid points. We average over  $10^5$  simulations on a  $30 \times 30$  grid and discs with radius  $r = 5$ . We indeed see that the boundary effects decay rapidly, after approximately one radius length. It also appears as though the boundary effects propagate like waves to the interior of the grid.

## 2 One-dimensional disc throwing

In this section, we analyse the one-dimensional restriction of the random disc thrower problem. This amounts to throwing equal-sized line segments on a line with  $n$  grid points. For simplicity, we shall usually assume that the segments all have radius  $r = 1$ , that is they each cover exactly three points of the line.

### 2.1 Recurrence

We now describe a recursive approach to the one-dimensional random disc thrower problem. Each time a disc is thrown, the problem naturally splits into two smaller subproblems which are independent of one another. See Figure 5. This leads to the

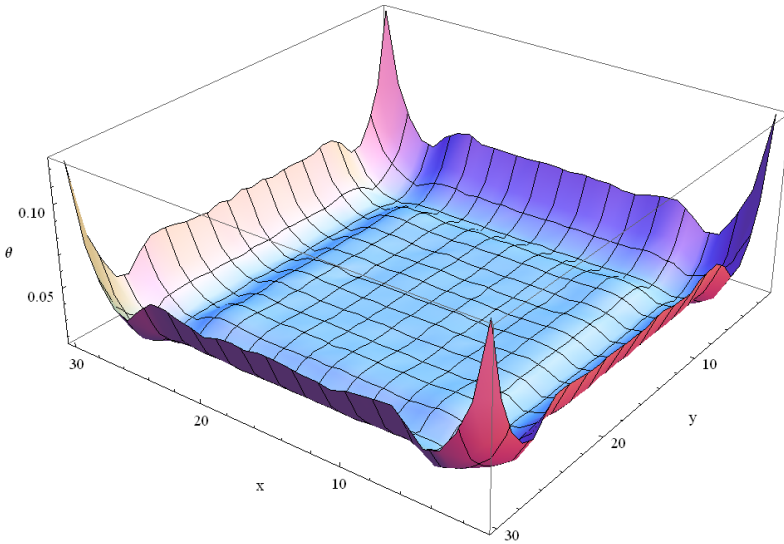


Figure 4: Simulation results of boundary effects.

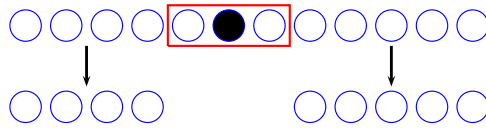


Figure 5: A depiction of the decomposition of the one-dimensional problem into two smaller independent subproblems.

following recursion:

$$\theta_1(n, 1) = 1 + \frac{1}{n} \sum_{k=3}^{n-2} (\theta_1(k-2, 1) + \theta_1(n-k-1, 1)) + \frac{2}{n} (\theta_1(n-3, 1) + \theta_1(n-2, 1)). \tag{1}$$

We note that the last term on the right hand side corresponds to the case where the disc has been thrown close to one of the endpoints of the line. Though the above recurrence is written only for  $r = 1$ , this relation can be extended to arbitrary  $r$ . Hence for reasonably small  $n$  (and  $r$ ) we can use the recurrence to compute  $\theta_1(n, r)$  exactly.

To compute the limit of  $\theta_1(n, 1)$ , we require analytical tools. This is done explicitly in Dutour Sikirić and Itoh [3, Chapter 2] for a closely related one-dimensional discrete random sequential packing problem, referred to as the Flory model. This process is

defined as follows: place at random a left-closed, right-open interval of length  $r$  in  $\{0, \dots, n\}$ , with the left endpoint of the interval chosen uniformly from  $\{0, \dots, n-r\}$ ; then each subsequent interval (of length  $r$ ) is chosen uniformly at random so that it intersects with no previously chosen interval. Here the problem is to study  $v_r(n)$ , the number of intervals that have been packed by the end of the process, and in particular to determine  $\mathbb{E}v_r(n)/n$ .

The relationship between the one-dimensional random disc thrower problem and the one-dimensional discrete random sequential packing is, for instance for  $r = 1$ , that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}v_1(n)}{n} = \liminf_{n \rightarrow \infty} \theta_1(n, 1). \quad (2)$$

It is possible to derive from a recurrence relation similar to ours in (1) a differential equation for the generating function

$$F(x) := \sum_{n=1}^{\infty} \frac{\mathbb{E}v_1(n)}{n} x^n.$$

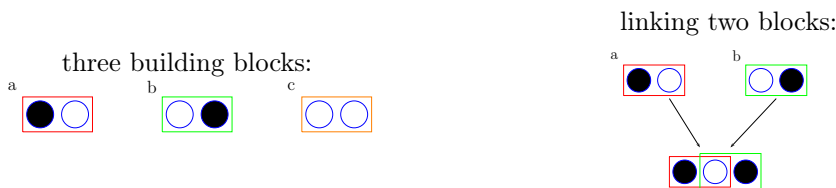
By solving the differential equation, we can then obtain an analytic expression for  $F(x)$ . Then reading off the coefficient of  $x^n$  in the Taylor series of  $F(x)$ , we obtain

$$\lim_{n \rightarrow \infty} \theta_1(n, 1) = \frac{1}{2} \left( 1 - \frac{1}{e^2} \right) \approx 0.432$$

where we used (2).

## 2.2 A Markov chain approximation

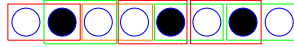
The following approach is suitable for the coverage for a very large grid compared to the size of the disc. Consider the one-dimensional case and a new random process with the following three building blocks, named  $a$ ,  $b$  and  $c$ . Linking two building blocks we require the endpoints to overlap and to be either both occupied or both empty. The following pictures illustrate the building blocks, how to link them, and give an example of an admissible configuration.



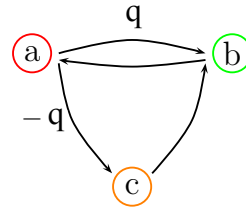
In order to approximate the one-dimensional random disc thrower problem, we would like to exclude on one hand the possibility of two centres being neighbouring points, and on the other hand the appearance of three (or more) unoccupied neighbouring grid points. Therefore, we consider the following transition matrix:



example of an admissible configuration:



$$\mathbb{M} := (\mathbb{M}_{ij})_{1 \leq i, j \leq 3} = \begin{pmatrix} 0 & q & 1 - q \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$



for  $0 < q < 1$ ,  $q$  being the only parameter that determines the Markov chain. Which value of  $q$  should we choose? Our Ansatz is the following: we choose the value of  $q$  to maximise the entropy of the system.

The entropy of the Markov chain with transition matrix  $\mathbb{M}$  is given by the entropy rate

$$\mathcal{H} = - \sum_{i,j} \pi_i \mathbb{M}_{ij} \log \mathbb{M}_{ij} = - \frac{1}{3-q} (q \log q + (1-q) \log(1-q)), \quad (3)$$

where  $\Pi = (\pi_1, \pi_2, \pi_3)$  with  $\pi_1 + \pi_2 + \pi_3 = 1$  is the stationary distribution, i.e. it satisfies the equation  $\Pi \mathbb{M} = \Pi$ . The latter two equations imply  $\Pi = (\frac{1}{3-q}, \frac{1}{3-q}, \frac{1-q}{3-q})$ . Note that  $\frac{1}{3-q}$  is an approximation for the coverage ratio  $\theta_1(1)$ .

Let us heuristically explain our Ansatz: the system is very likely not to be in an extreme event. A likely event is a steady state, i.e. a configuration that does not change much under small fluctuation. Entropy is the measure of the multiplicity of a configuration. So we can reformulate our task as to extremise the entropy. This Ansatz is consistent with the Second Law of Thermodynamics. The concept of entropy and the entropy maximisation principle also plays an essential role in information theory, see Cover and Thomas [1].

The mathematical justification is based on large deviations. Morally, we say that a sequence of random variables  $Y_n$  taking value on a (Polish) space  $Y$  satisfies a large deviation principle with a rate function  $I: Y \rightarrow [0, +\infty]$  if for any event  $A$

$$\mathbb{P}(Y_n \in A) \approx \exp[-n \inf_{x \in A} I(x)] \text{ as } n \rightarrow \infty.$$

The rate function  $I$  characterises the probability of observing an event: for an event  $A$ , the smaller values of  $\inf_{x \in A} I(x)$  yield higher probabilities of observing  $A$ . The value of  $I$  is always non-negative and attains its minimum at the most probable event. Thus the large deviation principle explains that in order to have the most likely event, we need to extremise the rate function.

Now we consider the empirical pair measure of an irreducible Markov chain  $X =$

$\{X_i\}$  where  $X_i \in \Gamma, i = 1, \dots, k$ ,

$$L_n = \frac{1}{n} \sum_{i=1}^n \delta_{(X_i, X_{i+1})}.$$

Sanov's Theorem (cf. den Hollander [2]) states that  $L_n$  satisfies a large deviation principle with the rate function

$$I_{\mathbb{M}}(\nu) = \sum_{ij} \nu_{ij} \log \left( \frac{\nu_{ij}}{\bar{\nu}_i \mathbb{M}_{ij}} \right)$$

for any probability measure  $\nu$  on  $\Gamma \times \Gamma$ , where  $\bar{\nu}_i = \sum_j \nu_{ij}$ . Since the empirical measure counts the (average) number of realisations of an event, the theorem says that in order to have a maximum number of realisation of a Markov chain, we should extremise the relative entropy. Note that we can rewrite  $I$  as a relative entropy

$$I_{\mathbb{M}}(\nu) = \mathcal{R}(\nu || \bar{\nu} \otimes \mathbb{M}), \quad (4)$$

where  $\bar{\nu} \otimes M$  is defined by  $(\bar{\nu} \otimes M)_{ij} = \bar{\nu}_i \mathbb{M}_{ij}$ . Now we can rewrite the function  $\mathcal{H}$  in (3) in the form of (4). Let  $u_{ij} = \frac{1}{|\Gamma|^2}$  be the uniform measure on  $\Gamma \times \Gamma$ .

$$\begin{aligned} \mathcal{H} &= - \sum_{i,j} \pi_i \mathbb{M}_{ij} \log \mathbb{M}_{ij} \\ &= - \sum_{ij} \pi_i \mathbb{M}_{ij} \log \frac{\pi_i \mathbb{M}_{ij}}{u_{ij}} + \log |\Gamma|^2 + \sum_i \pi_i \log \pi_i \\ &= -\mathcal{R}(\pi \otimes \mathbb{M} || u) + \log |\Gamma|^2 + \sum_i \pi_i \log \pi_i. \end{aligned}$$

Hence, up to sum of a constant and entropy of the stationary distribution,  $\mathcal{H}$  can be written as the (negative) relative entropy. The argument above explains why it is reasonable to maximise the function  $\mathcal{H}$ .

Therefore, we are finding the value  $q$  that satisfies

$$\frac{d\mathcal{H}}{dq} = 0.$$

Using (3), we can calculate explicitly

$$\frac{d\mathcal{H}}{dq} = - \frac{3 \log q - 2 \log(1-q)}{(3-q)^2},$$

that yields the equation  $q^3 - q^2 + 2q - 1 = 0$ . Solving this equation, we find  $q = 0.5698$  and  $\theta_1(1) \approx \frac{1}{3-q} = 0.4115$ .

This is close to the actual value ( $\approx 0.432$ , as we saw in the last subsection). We have hope to find better estimates of the coverage ratio by including longer building blocks and consequently studying larger transition matrices. This is a task for future research.

## 2.3 Configurations and weights

A final approach to understand the disc throwing problem is by an analysis of the direct calculation. Such a direct calculation of all possible disc throwings quickly becomes unfeasible when the number of grid points  $n$  is large compared to the disc size  $r$ . However, when we approach the problem from the end result of a random disc throwing, we see that the disc throwing process can be separated into two problems:

1. determine all possible configurations  $c_i$  of covering the grid.
2. determine the relative weight, i.e. the probability  $P_i$  of reaching a given configuration  $c_i$  by the disc throwing process.

Here  $i \in \{1, \dots, l\}$  runs over all possible configurations of covering the grid. If both problems are solved exactly, the expected value for the number  $N_1(n, 1)$  of discs to cover the grid equals

$$\mathbb{E}N_1(n, 1) = \sum_{i=1}^l k_i P_i, \quad (5)$$

where  $k_i$  is the number of discs in configuration  $c_i$ .

The advantage of separating the original disc throwing process in this way, is that the two subproblems might be easier to calculate or estimate than the disc throwing process itself.

### 2.3.1 Configurations

Determining all possible configurations of a completely covered grid of length  $n - 1$  ( $n$  grid points) by discs of size  $r = 1$  is not hard in the one-dimensional situation. For this, we have to fill an interval of length  $n - 1$  or  $n - 2$  or  $n - 3$  with intervals of length 2 or 3 and add 0, 1 or 2 intervals of length 1 at the boundary respectively to end up with a combined length equal to  $n - 1$ . These intervals represent the possible distances between neighbouring centers of discs. The following example with  $n = 21$  grid points (Table 2.3.1) should clarify the methodology.

$C$	$k_C$	similarity class of configurations	$a_C$
1	11	$10 \cdot 2$	$\binom{10}{0}$
2	10	$1 + 9 \cdot 2 + 1$	$\binom{9}{0}$
3	10	$1 + 8 \cdot 2 + 1 \cdot 3$ or $8 \cdot 2 + 1 \cdot 3 + 1$	$2 \cdot \binom{9}{8}$
4	10	$7 \cdot 2 + 2 \cdot 3$	$\binom{9}{7}$
5	9	$1 + 6 \cdot 2 + 2 \cdot 3 + 1$	$2 \cdot \binom{8}{6}$
6	9	$1 + 5 \cdot 2 + 3 \cdot 3$ or $5 \cdot 2 + 3 \cdot 3 + 1$	$2 \cdot \binom{8}{5}$
7	9	$4 \cdot 2 + 4 \cdot 3$	$\binom{8}{4}$
8	8	$1 + 3 \cdot 2 + 4 \cdot 3 + 1$	$\binom{7}{3}$
9	8	$1 + 2 \cdot 2 + 5 \cdot 3$ or $2 \cdot 2 + 5 \cdot 3 + 1$	$2 \cdot \binom{7}{2}$
10	8	$1 \cdot 2 + 7 \cdot 3$	$\binom{7}{1}$
11	7	$1 + 0 \cdot 2 + 6 \cdot 3 + 1$	$\binom{6}{0}$

Table 1: List of all possible similarity classes.  $n = 21$  and  $r = 1$ .

In the enumeration of possible configurations, we can restrict ourselves to enumerating different *similarity classes*  $C$ , due to the symmetry of the  $c_i$ . A similarity class  $C$  contains  $a_C$  configurations  $c_i$ , for which the analysis of the disc throwing process does not depend on the precise ordering of the different intervals of length 2 and 3.  $k_C$  stands for the number of discs used for each configuration in the similarity class  $C$ . As an example, the 5<sup>th</sup> similarity class consists of configurations with 6 intervals of length 2 and 2 intervals of length 3. A representative configuration from this class is presented in Figure 6. Since the probability  $P_i$  of one representative  $c_i$  in a similar-

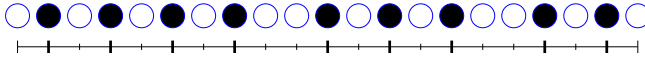


Figure 6: Representative configuration for  $C = 5$ .

ity class  $C$  equals the probability of any other representative in that class, the total probability of ending up in any configuration in a certain similarity class is  $a_C P_i$ .

### 2.3.2 Weights

**First approximation** The probabilities of occurrence for intervals of length 2 and 3 are not the same. From Monte Carlo simulations, we know that the probability of an interval of length 2 is higher than 0.5 and is length-of-grid-dependent. However, we can conclude that the error we make when we do not use the exact  $P_i$ , but rather assume that each configuration  $c_i$  is equally probable, is not large. In Figure 7, one can see the distribution of  $N_1(100, 1)$  under assumption that each configuration  $c_i$  is equally probable.

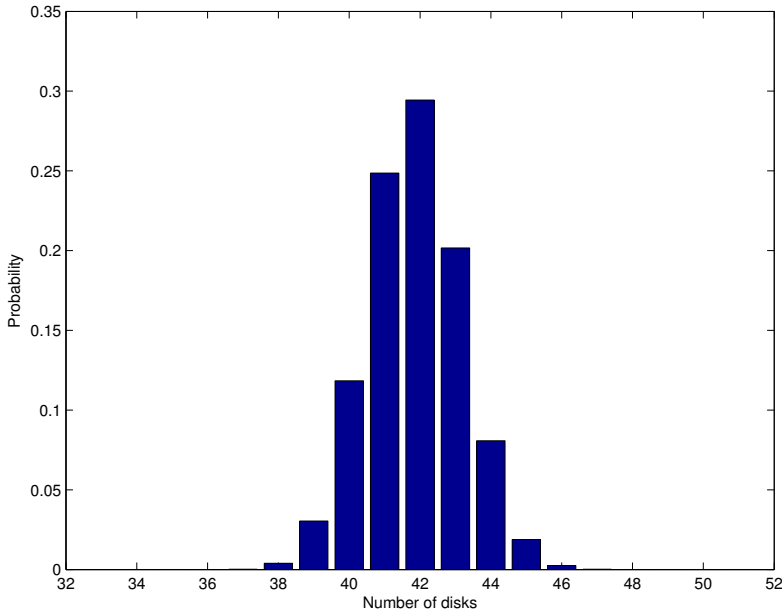


Figure 7: Distribution of discs needed to cover  $n = 100$  grid points.

We obtained  $\mathbb{E}N_1(100, 1) \approx 41.8370$ ,  $\text{Var } N_1(100, 1) \approx 1.7773$  and  $\sum_{i=1}^{N=51} a_i = 2066337330754$ . This result is not far from 10000 Monte Carlo simulations where we obtained  $\mathbb{E}N_1(100, 1) \approx 43.5225$ ,  $\text{Var } N_1(100, 1) \approx 1.9155$ . It would be useful to improve our results by approximating the  $P_i$ . The following section covers this subject.

**Approximation of  $P_i$**  Finding the probability  $P_i$  of ending up in a given configuration  $c_i$  brings us back to the original problem. If the number of discs in the configuration  $c_i$  is denoted by  $k_i$ , then there are  $k_i!$  ways of ending up in this configuration. At this stage, we want to emphasise that the probability is *not* given by the ratio of multiplicities of the different configurations,  $P_i \neq \frac{k_i!}{\sum_i k_i!}$ . Rather, to find  $P_i$  one has to consider each  $j \in \{1, \dots, k_i\}$  of the  $k_i!$  possible ways to end up in  $c_i$ . Each of these disc throwings depends on the order of the discs being thrown and how many grid points were covered in previous steps,

$$P_{i_j} = \prod_{m=1}^{k_i} \frac{1}{n - \sum_{s=0}^{m-1} z_{i_{j_s}}}, \quad (6)$$

where  $z_{i_{j_s}}$  is the number of additional grid points covered by the  $s$ 'th disc thrown in the  $j$ 'th process of covering configuration  $c_i$  and  $z_{i_{j_0}} = 0$ . Note that  $\sum_{s=1}^{k_i} z_{i_{j_s}} = n$ , since the grid is fully covered after all discs have been thrown.

Calculating  $P_{i_j}$  for all these different disc throwing processes is exactly what makes the disc throwing process hard. Therefore, in this method we suggest to approximate the additional coverings  $z_{i_{j_s}}$  in a sensible manner, based on what typically happens in a disc throwing process. As evidenced by Figure 8, the values for  $z_{i_{j_s}}$  are typically  $2r + 1 = 3$  for the first discs thrown. Over the course of the disc throwing process, some discs only cover 2 additional points, while at the end, most newly thrown in discs cover only a single additional point. On average, 0.5 of the discs cover 3 points,



Figure 8: Additional points covered by newly thrown in discs, for an  $n = 1000$  simulation.

0.31 of the discs cover 2 points and 0.19 of the discs cover 1 point.

As an approximation for  $P_{i_j}$  we assume that *all* initial discs cover 3 points, *all* next discs cover 2 additional points and *all* remaining discs cover a single point. Then,  $P_{i_j} = P_{i_1}$  for all  $j$  and we do not have to worry about different throwing processes. Denote with  ${}_p k_i$  the number of discs that cover  $p$  previously uncovered points, then from

$${}_3 k_i + {}_2 k_i + {}_1 k_i = k_i, \quad 3{}_3 k_i + 2{}_2 k_i + {}_1 k_i = n, \quad (7)$$

we can solve  ${}_2 k_i$  and  ${}_1 k_i$  in terms of  ${}_3 k_i$ . As a further approximation, we assume that  $\frac{{}_2 k_i}{{}_1 k_i} = \frac{0.31}{0.19}$ . This then fixes  ${}_3 k_i$  and allows us to write an approximate value for  $P_i$  as

$$P_i = k_i! \prod_{m=1}^{{}_3 k_i} \frac{1}{n - 3(m-1)} \times \prod_{m=1}^{{}_3 k_i} \frac{1}{n - 3{}_3 k_i - 2(m-1)} \\ \times \prod_{m=1}^{{}_1 k_i} \frac{1}{n - 3{}_3 k_i - 2{}_2 k_i - (m-1)}. \quad (8)$$

As a result of approximating the probabilities, we are not guaranteed to have a total probability equal to unity. These  $P_i$  should therefore be seen as *relative* probabili-

ties and, to find the expectation  $\mathbb{E}N_1(n, 1)$ , one should use the rescaled probabilities  $\hat{P}_i = P_i / \sum_{i=1}^l P_i$ . By this method, we find approximations  $\mathbb{E}N_1(21, 1) \approx 9.0749$  and  $\text{Var } N_1(21, 1) \approx 0.24479$ , which are in reasonable but not excellent agreement with 10000 Monte Carlo simulations that suggest  $\mathbb{E}N_1(21, 1) \approx 9.3769$  and  $\text{Var } N_1(21, 1) \approx 0.43549$ .

### 3 Two-dimensional disc throwing

In this section, we analyse the original two-dimensional random disc thrower problem. We have chosen to restrict our attention to discs of radius  $r = \sqrt{2}$ . This in fact corresponds to discs which are squares each covering 9 points of the grid.

#### 3.1 Recurrence approach

In Subsection 2.1, we outlined an explicit recurrence for the one-dimensional restriction of the problem. This recurrence relied on the fact that we could split into smaller independent subproblems. Here we would like to discuss the difficulties of extending this approach to the two-dimensional problem.

A natural idea is to split the problem at each stage into the four subproblems corresponding to the four regions of the grid determined by the randomly thrown disc, as depicted in Figure 9, left. Solving these as independent subproblems and then combining them provides a lower bound to a related disc throwing problem. We can solve this related problem by writing the corresponding recurrence relation, one that is similar to (1) but with four terms rather than two. As in Subsection 2.1, such a recurrence allows us to write down exact answers for small values of  $n$ , or to potentially solve it asymptotically using analytic methodology.

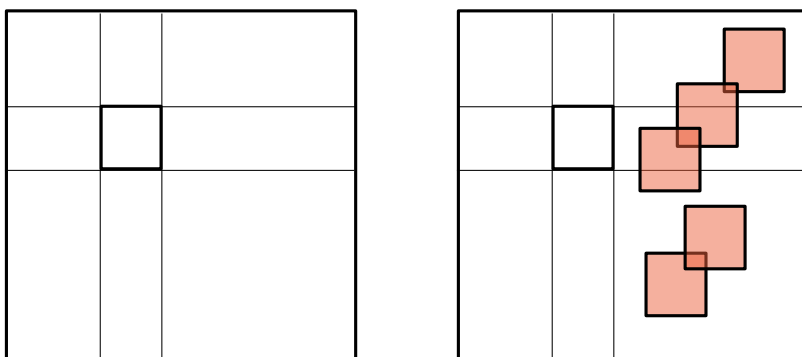


Figure 9: A depiction of the problem with decomposition of the two-dimensional problem into smaller subproblems.

Unfortunately, in attempting to decompose the problem in this way, there are possible long-range dependencies between the four regions, as depicted in Figure 9, right. Though it may be possible to generate high quality solutions for the related problem, it would require further investigation to determine if such solutions are at all related to the original problem. In particular, it would be worthwhile quantifying the error due to long-range dependencies. Perhaps they are negligible when  $n$  is large.

### 3.2 Markov chain approach

In two dimensions the Markov chain approach could be extended to the consideration of *Markov fields*, see e.g. [5]. Let us first recall the definition of a Markov random field on a regular two-dimensional grid.

Let  $S = \{1, \dots, n\} \times \{1, \dots, n\}$  be the grid of  $n^2$  points, which we call sites. For a fixed site  $s$  define its neighborhoods  $N(s)$ . For instance, for the site  $s = (i, j)$ , that is in the interior, the neighborhood could be  $N(s) = \{(i-1, j), (i+1, j), (i, j-1), (i, j+1)\}$ . For a site on the boundary, there are less neighbors.

A Markov random field  $X(S)$  on  $S$  is defined via local conditionals

$$\mathbb{P}(X(s) = x_s \mid X(S \setminus s) = x_{S \setminus s}) = \mathbb{P}(X(s) = x_s \mid X(N(s)) = x_{N(s)}).$$

In other words, the full conditional distribution of  $X(s)$  depends only on the neighbors  $X(N(s))$ .

To generalise our Markov approach to the two-dimensional case, we observe that the probability of a point in the lattice chosen to be a centre given all other outcomes is the same as the probability of being a centre conditioned only on the outcome of the neighbors. We see that the space of configurations as well as the above requirement in the two-dimensional disc thrower problem resemble with the definition of the Markov random field on the grid. For this reason, we believe that the Markov random field would be a good model for the two-dimensional disc thrower problem. The study of this model is out of the scope of this report and is a topic for future research.

### 3.3 Configurations and weights approach

Extending the approach of Subsection 2.3 to more than one dimension introduces additional complications. The feasibility of that method is determined by how well one can recover from the complications in each of the two subproblems.

**Configurations** Finding all configurations in a two-dimensional grid is hardly as straightforward as it is in the one-dimensional case. At least the separation of the disc throwing process into the two subproblems (finding configurations and weights) allows us to first only consider the problem of finding all possible configurations rather than all disc throwing processes. In principle, all configurations can be algorithmically enlisted by the following method: for a grid covered by  $r = \sqrt{2}$  discs, we know that exactly one of the 4 upper left grid points should be covered. Depending on which grid point happens to be covered, there are again 4 possibilities for the disc to the



right of it. By continuing this process also in the vertical direction, one finds all configurations. However, for an  $n \times n$  grid, there are a maximum of  $\lceil n/2 \rceil^2$  discs, each with at most 4 possible center points, giving an upper bound for the number of configurations equal to  $2^{2\lceil n/2 \rceil^2}$ . To get a feeling for the number of configurations, taking square discs covering 25 grid points on a  $30 \times 30$  lattice, there are about  $9^{\lceil 30/3 \rceil^2} \approx 10^{95}$  configurations.

**Weights** Extending our approximation of the probability  $P_i$  of a given configuration to the two-dimensional case does not seem to be the weakest link in this approach. Once the method has been fine tuned in the one-dimensional case, one can generalise easily to the two-dimensional situation. The probability  $P_i$  only depends on the number of discs  $k_i$  in the configuration, leading to only a small number of required calculations. The difficulty is how to find reasonable values for the  ${}_p k_i$ . In the one-dimensional situation discussed above, only  ${}_3 k_i$  had to be determined by the assumption that  $\frac{{}_2 k_i}{{}_1 k_i} = \frac{0.31}{0.19}$ . In two dimensions, there are many more  ${}_p k_i$  and still only two constraint equations  $\sum_p {}_p k_i = k_i$  and  $\sum_p p {}_p k_i = n^2$ . However, this is not a problem inherent to the two-dimensional situation, as it also occurs in the one-dimensional case for  $r > 2$ . Therefore, an analysis of the applicability of our approximation for the probabilities in this more general one-dimensional situation, should also be conclusive for the two-dimensional system.

### 3.4 Hexagonal approach

One can consider other lattices than the square one. One of the most common lattices in nature is the hexagonal one. While in a square lattice every point can be covered up to four times, in the hexagonal lattice every disc (with  $r = 1$ ) covers up to seven points and every point can be covered at most three times. This may simplify matters. Figure 10 depicts one of the possible configurations for a hexagonal lattice. Determining the expected number of randomly thrown discs needed for coverage of a given region is a problem that remains out of reach for the hexagonal lattice, yet we can still make some comparisons.

We have compared the square and hexagonal lattices directly, by performing 10000 Monte Carlo simulation on a  $20 \times 20$  grid. Table 2 presents the outcomes.

Lattice	$\mathbb{E}N_2(20, 1)$	$Var N_2(20, 1)$	$\theta$	area
square	149.1485	22.6434	0.3729	400
hexagonal	97.8310	7.4639	0.2824	$\approx 346.41$

Table 2: Comparison of characteristics of  $N_2(20, 1)$  for a square and hexagonal lattice for a  $20 \times 20$  grid.

On the one hand, with 400 points in a hexagonal lattice we can cover an area which is approximately 12.5% smaller than an area covered in a square fashion. On

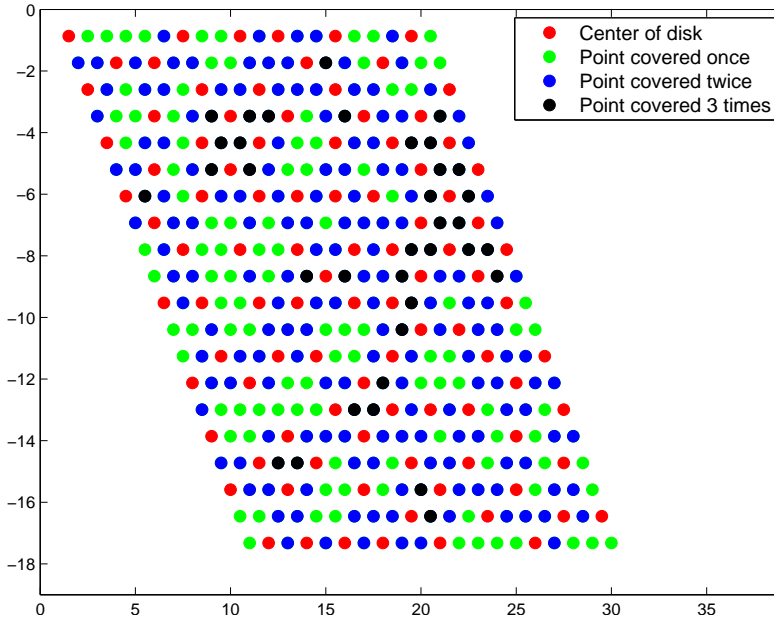


Figure 10: One realisation of a coverage process for a hexagonal lattice in a  $20 \times 20$  grid.

the other hand, we need 34% fewer discs and the coverage is 24% smaller, which makes the hexagonal approach interesting for further investigations.

## 4 The second problem of Philips

As a second question, we shortly discuss a related but different disc throwing process that models information propagation on a lattice. We restrict ourselves to a one-dimensional grid with  $n$  grid points. In this second problem, at a given intermediate stage, the first  $m$  grid points serve as possible locations for the centre of a disc. Let us call these  $m$  grid points the *base for disc throwing*. During 1 *uts* (unit time step) the usual disc throwing process is applied to this base for disc throwing. After all discs have been thrown,  $m'$  discs will be covered, where  $m' \geq m$  depending on the position of the (right) boundary disc. For the next *uts* the usual disc throwing process is applied to the new base for disc throwing, consisting of the  $m'$  grid points that were covered in the previous phase. The process continues until all  $n$  grid points are covered. The question is: how long, i.e. how many unit time steps, will it take on average to cover a grid of  $n$  grid points when one starts by throwing a disc centred on the first (left) grid point?

We consider discs of radius  $r = 1$ . At an intermediate stage — when the base

for disc throwing consists of  $m$  points — the signal is only propagated when the disc throwing process produces a configuration in which there is a disc centred on the last allowed grid point  $m$ , thereby expanding the base for disc throwing for the next phase to  $m + 1$ . Instead, if there is a disc centred at  $m - 1$ , there will be no additional points covered and the signal will not have propagated during this unit time step. With  $P(k, m)$  we denote the probability that at the end of the usual disc throwing process on the base of  $m$  points, there is a disc centred at the  $k$ 'th point. Note that  $P(m - 1, m) + P(m, m) = 1$ . The average speed of propagation is  $v(m) = (0 \cdot P(m - 1, m) + 1 \cdot P(m, m)) / (1 \text{ uts}) = P(m, m) / (1 \text{ uts})$  when the base for disc throwing has size  $m$ . Hence, on average the time it takes to expand to a base of size  $m + 1$  is  $t(m) = \frac{1}{1 - P(m-1, m)} \text{ uts}$  and the total time it takes to propagate information along a grid of size  $n$  is  $T(n) = \sum_{m=1}^{n-1} t(m) \text{ uts}$ .

For small values of  $m$ , the probability  $P(m - 1, m)$  is quickly calculated:  $P(0, 1) := 1 - P(1, 1) = 0$ ,  $P(1, 2) = \frac{1}{2}$ ,  $P(2, 3) = \frac{1}{3}$ . To find a closed expression for  $P(m - 1, m)$  for arbitrary  $m$ , we refer to ([4], page 1286) in which a problem isomorphic to the one presented here is studied. A recursive formula for  $P(m - 1, m)$ ,

$$P(m - 1, m) = \frac{1}{m} (P(m - 3, m - 2) + P(m - 4, m - 3) + \dots + P(2, 3) + P(1, 2) + 1), \quad (9)$$

follows from adding probabilities corresponding to the grid sizes to the right of the first disc thrown. This recursive relation is solved by  $P(m - 1, m) = \sum_{r=0}^m \frac{(-1)^r}{r!}$ , which approaches  $e^{-1}$  for large  $m$ . Hence, the average time of propagating through a lattice of  $n$  grid points is

$$T(n) = \sum_{m=1}^{n-1} \frac{1}{1 - \sum_{r=0}^m \frac{(-1)^r}{r!}} \text{ uts}, \quad (10)$$

which is easy to compute for small  $n$ , e.g.  $T(10) = 14.007 \text{ uts}$ . Furthermore, because  $P(m - 1, m)$  approaches  $e^{-1}$  rapidly for large  $m$ , we can find a linear propagation formula,  $T(n) \approx 14.007 + 1.582(n - 10) \text{ uts}$ , for  $n \geq 10$ .

In this analysis, we have only considered discs of radius  $r = 1$ . Our methods can be generalised to larger discs,  $r \geq 2$ . The average propagation velocity is then given by  $v(m) = (P(m - r + 1, m) + 2P(m - r + 2, m) + \dots + rP(m, m)) / (1 \text{ uts})$  and each of the  $P(k, m)$  are given in terms of a recursive formula. Extending the one-dimensional situation to a two-dimensional setup requires more care, however. In two dimensions, more exotic topologies are possible and the expansion rate is a more subtle notion to define. Nevertheless, the one-dimensional situation described in this section is already interesting by itself for its corridor or street/highway applications.

## References

- [1] T. M. Cover and J. A. Thomas. *Elements of information theory*. Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, second edition, 2006. ISBN 978-0-471-24195-9; 0-471-24195-4.

- 
- [2] F. den Hollander. *Large deviations*, volume 14 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 2000. ISBN 0-8218-1989-5.
- [3] M. Dutour Sikirić and Y. Itoh. *Random sequential packing of cubes*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011. ISBN 978-981-4307-83-3; 981-4307-83-1. doi: 10.1142/9789814307840. URL <http://dx.doi.org/10.1142/9789814307840>.
- [4] J. Evans. Random and cooperative sequential adsorption. *Reviews of Modern Physics*, 65(4):1281, 1993.
- [5] R. Kindermann and J. L. Snell. *Markov random fields and their applications*, volume 1 of *Contemporary Mathematics*. American Mathematical Society, Providence, R.I., 1980. ISBN 0-8218-5001-6.
- [6] I. Palásti. On some random space filling problems. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 5:353–360, 1960.
- [7] M. D. Penrose. Random parking, sequential adsorption, and the jamming limit. *Comm. Math. Phys.*, 218(1):153–176, 2001. ISSN 0010-3616. doi: 10.1007/s002200100387. URL <http://dx.doi.org/10.1007/s002200100387>.
- [8] M. D. Penrose. Limit theorems for monotonic particle systems and sequential deposition. *Stochastic Process. Appl.*, 98(2):175–197, 2002. ISSN 0304-4149. doi: 10.1016/S0304-4149(01)00152-1. URL [http://dx.doi.org/10.1016/S0304-4149\(01\)00152-1](http://dx.doi.org/10.1016/S0304-4149(01)00152-1).
- [9] A. Rényi. On a one-dimensional problem concerning random space filling. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 3(no 1/2):109–127, 1958.
- [10] J. Wang. Series expansion and computer simulation studies of random sequential adsorption. *Colloids and Surfaces A: Physicochemical and Engineering Aspects*, 165(1):325–343, 2000.