The ING problem: a problem from the financial industry

Cornelis W. Oosterlee[*]

In the 2007 Mathematics with Industry workshop, ING posed a challenging problem from financial mathematics. For a system of stochastic differential equations, representing an advanced model for asset prices, the question was whether a closed, or semi-closed, form of a pricing formula for call options could be derived. The asset price model of interest was the so-called hybrid Heston–Hull–White model.

The industrial interest comes from the fact that valuing and risk-managing derivatives demands fast and accurate prices. As the financial models used in practice are becoming increasingly complex, efficient solution methods have to be developed to cope with such models. Needless to say that working with a closed form solution is highly efficient.

The basis of modern option pricing theory is found in the famous Black–Scholes model, which itself is based on a one-factor stochastic model for asset prices,

$$dS_t = rS_t dt + \sqrt{v} S_t dW_t.$$

Here S_t denotes asset price, and W_t denotes Brownian motion. Interest rate r and 'volatility' $\sqrt{v}$ are assumed to be constant in this model which is a major model simplification. Based on this model derivatives, like options, can be priced highly efficiently.

The motivation behind using more general processes is the simple fact that the Black–Scholes model is not able to reproduce important empirical features of asset returns and at the same time provide a reasonable fit to the so-called implied volatility surfaces observed in option markets. Over the past few years it has been shown that several models that incorporate stochastic volatility are, at least to some extent, able to reproduce the volatility skew or smile. The particular model we will consider here is a more advanced form of the well-known Heston stochastic volatility model. The model is a generalization as also the interest rate is modeled by a stochastic differential equation. The hybrid Heston–Hull–White asset price model reads:

$$\begin{cases} dS_t &= r_t S_t dt + \sqrt{v_t} S_t dW_{1,t}, \\ dv_t &= \kappa(\eta - v_t)dt + \lambda \sqrt{v_t}\, dW_{2,t}, \\ dr_t &= (\theta(t) - a r_t)dt + \sigma dW_{3,t} \end{cases}$$

for $0 \le t \le T$ with T the maturity of the option. Here S_t, v_t, r_t denote the random variables asset price, its variance and interest rate, respectively, at time $t \ge 0$. The model constitutes an extension of the well-known Black–Scholes model as the volatility and the interest rate both evolve randomly over time. The quantities κ, η, λ, a, σ are positive real constants, that can be calibrated to market data. Furthermore, $\theta(t)$ is a deterministic, continuous, positive function of time which can be chosen

[*]CWI, Amsterdam, and Technische Universiteit Delft, editor ING problem, c.w.oosterlee@cwi.nl

as to match the so-called term structure of interest rates. Finally, $W_{1,t}$, $W_{2,t}$, $W_{3,t}$ denote Brownian motions with a positive covariance matrix

$$\mathrm{var}_{\mathbb{P}}(\tilde{\mathbf{W}}_t) := \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix} t.$$

By means of the risk-neutral valuation formula, the price of any European option can be written as an expectation of the discounted payoff of this option. Starting from this representation one can apply several techniques to calculate the price itself. Broadly speaking one can distinguish two types of methods: Solution of the corresponding partial differential equation (PDE) or stochastic differential equation (SDE) by integration. Both solution approaches may rely on techniques from numerical mathematics, including Monte Carlo simulation, in particular when pricing early exercise options or complex option contracts.

Quite a few mathematicians took up this ING challenge, and during the week three subgroups were formed, each approaching the problem from a different side. A particular challenge here was that some Dutch professors in financial mathematics in earlier attempts were not able to come up with a closed form option pricing solution for this particular model. It is therefore no surprise that the problem in its full generality could not be solved within the workshop week. However, three high-quality approaches with interesting insights are presented hereafter that allow for different dependency structures. We believe that based on the results in the contributions the pricing of options under the dynamics of the complete hybrid Heston–Hull–White model, such as by classical Monte Carlo simulation, can be significantly accelerated[1]

On behalf of the group participants we would like to thank in particular Dr. Antoine van der Ploeg from ING for his detailed technical note on the problem and for his assistance during the workshop.

[1] We would also like to point to work by N. Kunitomo and Y-J Kim, which can be found at http://www.e.u-tokyo.ac.jp/ kunitomo/Effects.pdf, which contains interesting aspects for our problem, but which we did not study during the workshop.

Three approaches to extend the Heston model

Michael Muskulus[*]

1 Introduction

The stock price in the Heston model [8] is given by the following stochastic differential equation

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sqrt{v_t}S_t\mathrm{d}W_{1,t}, \quad S_0 > 0,$$

where $r > 0$ denotes the risk-free interest rate, which is assumed to be constant in time. Since S_t follows a geometric Brownian motion, it is advantageous to consider $X_t = \ln S_t$ instead. By the Itô–Doeblin formula one then has

$$\mathrm{d}X_t = \mathrm{d}\ln S_t = (r - \frac{1}{2}v_t)\mathrm{d}t + \sqrt{v_t}\mathrm{d}W_{1,t}.$$

The volatility of the instantaneous stock returns $\mathrm{d}S_t/S_t$ follows the process

$$\mathrm{d}v_t = \kappa(\eta - v_t)\mathrm{d}t + \lambda\sqrt{v_t}\mathrm{d}W_{2,t}, \quad v_0 > 0,$$

in which $\kappa > 0$ determines the speed of adjustment of the volatility towards its theoretical mean $\eta > 0$, and $\lambda > 0$ is the second-order volatility, i.e., the variance of the volatility. Note that this has exactly the form as the Cox-Ingersoll-Ross (CIR) [6] interest rate process.

The money-market account evolves according to the ordinary differential equation $\mathrm{d}M_t = rM_t\mathrm{d}t$ with solution $M_t = M_0e^{rt}$. The importance of the Heston model comes from the fact that it allows for a semi-analytical solution in terms of characteristic functions (see Section 3).

2 Extension of the Heston model

Although the Heston model incorporates stochastic volatility, the fixed interest rate is an unrealistic assumption. Let us therefore consider (following [14]) a generalized Hull–White process [9] for the interest rate,

$$\mathrm{d}r_t = (\theta_t - ar_t)\mathrm{d}t + \sigma\mathrm{d}W_{3,t},$$

where $\theta_t > 0$, $t \in \mathbb{R}$, is a nonconstant drift term. Usually, stock rate, volatility, and interest rate are correlated; a phenomenon known as the leverage effect [2, 3]. Assume that

$$\mathrm{d}W_{i,t}\mathrm{d}W_{j,t} = \rho_{ij}\mathrm{d}t,$$

[*]Universiteit Leiden, muskulus@math.leidenuniv.nl

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where

$$C = (\rho_{ij})_{1\le i,j\le 3} = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{21} & 1 & \rho_{23} \\ \rho_{31} & \rho_{32} & 1 \end{pmatrix}$$

is a constant[1] covariance matrix, and therefore positive semi-definite. In fact, for the application in finance, we can assume that C is nonsingular[2].

From the spectral theorem of linear algebra we see that C, being positive definite and symmetric, has a unique matrix square root $A = (a_{ij})_{1\le i,j\le 3}$, such that

$$C = U\Sigma U^t = (U\Sigma^{1/2})(U\Sigma^{1/2})^t = AA^t, \tag{2.1}$$

where $U\Sigma U^t$ is the singular-value decomposition of C. Explicitly, we have

$$\sum_{k=1}^{3} a_{ik}a_{jk} = \rho_{ij}, \quad \text{for all } i, j = 1, 2, 3.$$

There now exist adapted, independent Brownian motions $B_{i,t}$, $i = 1, 2, 3$, such that

$$dW_{i,t} = \sum_{j=1}^{3} a_{ij}\,\mathrm{d}B_{j,t},$$

and the general model we consider here is the following:

$$\mathrm{d}S_t = r_t S_t \mathrm{d}t + \sqrt{v_t} S_t\, a_{1i}\mathrm{d}B_{i,t} \quad \text{or} \quad \mathrm{d}X_t = (r_t - \frac{1}{2}v_t)\mathrm{d}t + \sqrt{v_t} a_{1i}\mathrm{d}B_{i,t} \tag{2.2}$$

$$\mathrm{d}v_t = \kappa(\eta - v_t)\mathrm{d}t + \lambda\sqrt{v_t}\, a_{2j}\mathrm{d}B_{j,t} \tag{2.3}$$

$$\mathrm{d}r_t = (\theta_t - a r_t)\mathrm{d}t + \sigma\, a_{3k}\mathrm{d}B_{k,t}, \tag{2.4}$$

where the Einstein convention for summation of repeated indices is used. The money market account develops according to

$$M_t = M_0 \exp\left(\int_0^t r_s \mathrm{d}s\right).$$

In this generality, the model is probably not solvable (semi-) analytically. Therefore three different constraints, arising from different strategies are discussed that lead to partial solutions.

3 Independent interest process

The first simplification is to assume that the interest rate process r_t evolves independently from the stock price and volatility processes S_t and v_t, keeping the correlation between the latter two,

$$\mathrm{d}W_{1,t}\mathrm{d}W_{2,t} = \rho \mathrm{d}t$$
$$\mathrm{d}W_{1,t}\mathrm{d}W_{3,t} = \mathrm{d}W_{2,t}\mathrm{d}W_{3,t} = 0.$$

[1] The decomposition of correlated Brownian motions into independent ones we are about to describe is also possible if $C = C(t)$ is an adapted process in time.

[2] This is possible since we will never have a perfectly linear relation between the driving Brownian motions of stock price, volatility, and interest rate — this would be rather contradictory to the assumption of stochasticity, and in such a case we could do with a simpler model than the one considered.

The first relation can be rewritten[3] as

$$\mathrm{d}W_{1,t} = \rho \mathrm{d}W_{2,t} + \sqrt{1-\rho^2}\mathrm{d}W'_{2,t},$$

where $W'_{2,t}$ is another Brownian motion, independent of $W_{2,t}$.

Define the integrated interest $R_t = \int_0^t r_t \mathrm{d}t$. We want to find the European call option price at maturity time T, given an initial stock price S_0, volatility v_0 and interest rate r_0 (and initial time $t = 0$),

$$\begin{aligned} C_T(S_0, v_0, r_0) &= \mathbb{E}[e^{-R_T}(S_T - K)^+ \mid S_0, v_0, r_0] \\ &= \mathbb{E}[e^{-R_T} S_T \cdot 1_{(\ln S_T > \ln K)}] - K\mathbb{E}[e^{-R_T} \cdot 1_{(\ln S_T > \ln K)}] \\ &= \mathbb{E}[e^{-R_T} S_T] \frac{\mathbb{E}[e^{-R_T} S_T \cdot 1_{(\ln S_T > \ln K)}]}{\mathbb{E}[e^{-R_T} S_T]} - K\mathbb{E}[e^{-R_T}] \frac{\mathbb{E}[e^{-R_T} \cdot 1_{(\ln S_T > \ln K)}]}{\mathbb{E}[e^{-R_T}]}, \end{aligned}$$

where $x^+ = \max(0, x)$ denotes the positive part of x, and 1_A is the indicator function of the event A. Note that under the risk-neutral measure the process $(e^{-R_t} S_t)_{t \geq 0}$ is a martingale, such that $\mathbb{E}[e^{-R_t} S_t] = S_0$.

Define an (analytic) function

$$\Psi(z) = \mathbb{E}[e^{-R_T + z \ln S_T}], \quad z \in \mathbb{C},$$

such that

$$\Psi(0) = \mathbb{E}[e^{-R_T}] = P(r_0, T)$$

is the discount price function, i.e., the price of a zero-coupon bond at time T.

Consider now the two (scaled) characteristic functions

$$\begin{aligned} \Phi_1(z) &= \frac{\Psi(1+iz)}{\Psi(1)} = \frac{\mathbb{E}[e^{-R_T} S_T e^{iz \ln S_T}]}{\mathbb{E}[e^{-R_T} S_T]} \\ \Phi_2(z) &= \frac{\Psi(iz)}{\Psi(0)} = \frac{\mathbb{E}[e^{-R_T} e^{iz \ln S_T}]}{\mathbb{E}[e^{-R_T}]} \end{aligned}$$

for two distribution functions F_1, F_2.

The particular form of these functions is a consequence of the generalized Bayes theorem [12, pg. 231] for conditional expectations, when we require

$$C_T(S_0, v_0, r_0) = S_0 \int_{\ln K}^{\infty} \mathrm{d}F_1(x) - KP(r_0, T) \int_{\ln K}^{\infty} \mathrm{d}F_2(x). \tag{3.1}$$

Fourier inversion[4] then allows to numerically evaluate the probability distributions [4], such that

[3] This is nothing else than the two-dimensional analogue of the matrix square root decomposition, Eq. (2.1).

[4] The inversion formula goes back to Gurland [7], who showed that

$$F(x) + F(x-0) = 1 - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-iux}\Phi(u)}{iu} \mathrm{d}u,$$

where the integral has to be interpreted as a Cauchy principal value. For (left-) continuous $F(x)$ this reduces to

$$P(X \leq x) = F(x) = \frac{1}{2} + \frac{1}{2\pi} \int_0^{\infty} \frac{\Phi(-u)e^{iux} - \Phi(u)e^{-iux}}{iu} \mathrm{d}u,$$

such that

$$P(X \geq \ln K) = 1 - F(\ln K) = \frac{1}{2} - \frac{1}{2\pi} \int_0^{\infty} \frac{\Phi(-u)e^{iu \ln K} - \Phi(u)e^{-iu \ln K}}{iu} \mathrm{d}u.$$

the option pricing function at time t is

$$C_{T-t}(S_t, v_t, r_t) = S_t\left(\frac{1}{2} - \frac{1}{2\pi}\int_0^\infty \frac{\Phi_1(-u)e^{iu\ln K} - \Phi_1(u)e^{-iu\ln K}}{iu}\mathrm{d}u\right)$$
$$- KP(r_t, T-t)\left(\frac{1}{2} - \frac{1}{2\pi}\int_0^\infty \frac{\Phi_2(-u)e^{iu\ln K} - \Phi_2(u)e^{-iu\ln K}}{iu}\mathrm{d}u\right).$$

The remaining work is to find an expression for $\Psi(z)$. This method is due to Scott [11], and we just follow his calculations (see Appendix for details), to arrive at

$$\Psi(z) = e^{-z(v_0+\kappa\eta T)} \cdot \mathbb{E}[e^{(z-1)R_T}] \cdot \mathbb{E}[e^{wV_T + z\frac{\rho}{\lambda}v_T}], \tag{3.2}$$

where we used the integrated volatility $V_t = \int_0^t v_t \mathrm{d}t$, and

$$w = (z-1)z\frac{1}{2}(1-\rho^2) + z\left(\frac{\rho}{\lambda}\kappa - \frac{1}{2}\rho^2\right).$$

From the theory of Bessel bridges [10, 5] we have the following closed form for the second expectation:

$$\mathbb{E}[e^{-s_1V_T - s_2v_T} \mid v_0] = e^{a_T - b_Tv_0}, \qquad \text{Re } s_i \geq 0, \quad i = 1, 2,$$

where

$$a_T = 2\kappa\eta \cdot \ln\frac{2\gamma e^{\frac{1}{2}(\kappa-\gamma)T}}{2\gamma e^{-\gamma T} + (\kappa+\gamma+s_2)(1-e^{-\gamma T})}$$
$$b_T = \frac{(1-e^{-\gamma T})(2s_1 - \kappa s_2) + \gamma s_2(1+e^{-\gamma T})}{2\gamma e^{-\gamma T} + (\kappa+\gamma+s_2)(1-e^{-\gamma T})},$$

and $\gamma = \sqrt{\kappa^2 + 2s_1}$. The parameters κ and η are taken from the volatility process:

$$\mathrm{d}v_t = \kappa(\eta - v_t)\mathrm{d}t + \lambda\sqrt{v_t}\mathrm{d}W_{2,t}, \quad v_0 > 0. \tag{3.3}$$

This almost solves the problem, since we still need to find an expression for the first expectation in Eq. (3.2).

If we now replace[5] the generalized Hull–White interest rate process with a CIR type interest process,

$$\mathrm{d}r_t = (\theta - ar_t)\mathrm{d}t + \sigma\sqrt{r_t}\mathrm{d}W_{3,t},$$

then this is also of the above form (3.3) (replacing κ by a, and η by θ/a), giving us a semi-analytical solution.

4 Constrained correlations

We now present an alternative method. Consider the model (2.2-2.4) again.

The change of variable $S_t = \exp(X_t)$ leads to $G_t(X_t, \cdot, \cdot, \cdot) = C_t(S_t, \cdot, \cdot, \cdot)$, such that $(e^{-R_t}G_t)$ is a martingale (under the appropriate, equivalent risk-neutral measure). Following the strategy of the

[5] In fact, it should be possible to arrive at a similar expression for the (standard) Hull–White interest rate process, too, by following the lines of the proof of above formula in [10, 5]. This is one possible direction for future research.

multi-dimensional Feynman-Kac theorem for independent Brownian motions [13], we expand the differential $d(e^{-R_t}G_t)$ in dt and $dB_{i,t}$ terms ($i = 1, 2, 3$), and set the dt term equal to zero, leading[6] to the following PDE:

$$\begin{aligned} r_t G_t = \frac{\partial G_t}{\partial t} + (r_t - \frac{1}{2}v_t)\frac{\partial G_t}{\partial X_t} + \kappa(\eta - v_t)\frac{\partial G_t}{\partial v_t} + (\theta_t - a r_t)\frac{\partial G_t}{\partial r_t} \\ + \frac{1}{2}v_t\frac{\partial^2 G_t}{\partial X_t^2} + \frac{1}{2}\lambda^2 v_t\frac{\partial^2 G_t}{\partial v_t^2} + \frac{1}{2}\sigma^2\frac{\partial^2 G_t}{\partial r_t^2} \\ + \lambda\rho_{12}v_t\frac{\partial^2 G_t}{\partial X_t \partial v_t} + \sigma\rho_{13}\sqrt{v_t}\frac{\partial^2 G_t}{\partial X_t \partial r_t} + \lambda\sigma\rho_{23}\sqrt{v_t}\frac{\partial^2 G_t}{\partial v_t \partial r_t}. \end{aligned}$$

The ansatz[7]

$$G_t = e^{A(T-t)+v_t B(T-t)+r_t C(T-t)+\sqrt{v_t}D(T-t)+iuX_t}$$

now gives[8] the following system of equations:

$$\begin{aligned} \frac{dA}{dt} &= \theta_t C(t) + \frac{1}{2}\sigma^2 C^2(t) + \kappa\eta B(t) + \frac{1}{2}\lambda\sigma\rho_{23}D(t)C(t) + \frac{1}{8}\lambda^2 D^2(t) \\ \frac{dB}{dt} &= -\frac{iu}{2} - \frac{u^2}{2} - \kappa B(t) + \lambda\rho_{12}iuB(t) + \frac{1}{2}\lambda^2 B^2(t) \\ \frac{dC}{dt} &= iu - aC(t) \\ \frac{dD}{dt} &= iu\sigma\rho_{13}C(t) - \frac{1}{2}\kappa D(t) + iu\frac{1}{2}\lambda\rho_{12}D(t) + \lambda\sigma\rho_{23}B(t)C(t) + \frac{1}{2}\lambda^2 B(t)D(t) \\ 0 &= 8D(t)\left(4\kappa\eta - \lambda^2\right) \end{aligned}$$

which is a system of ODEs, either (i) if we set

$$\lambda = 2\sqrt{\kappa\eta} \qquad \text{(Forced volatility variance)},$$

or (ii) if we set D(t) = 0. The latter is possible, if we let $B(t) = -iu\frac{\rho_{13}}{\rho_{23}}\frac{1}{\lambda}$, which gives us two constraints on the parameters (from $\frac{dB}{dt} = 0$):

$$\rho_{23} = \frac{2\kappa}{\lambda}\rho_{13}, \qquad \rho_{12} = \frac{4\kappa^2 + \lambda^2}{4\kappa\lambda} \qquad \text{(Forced volatility correlation)}.$$

In this case, the equation in $A(t)$ can be integrated easily, since $C(t)$ is readily available,

$$C(t) = \frac{iu}{a}\left(e^{at} - 1\right), \quad \text{when } C(0) = 0.$$

Furthermore, if θ_t is assumed constant, the solution is given analytically by the characteristic function of G_t, as in the solution of the Heston model.

[6]Note that $a_{ik}dB_{k,t} \cdot a_{jl}dB_{l,t} = a_{ik}a_{jl}\delta_{kl}dt = a_{ik}a_{jk}dt = \rho_{ij}dt$, where δ_{kl} is the Kronecker delta.

[7]Which fulfills the necessary boundary condition $G_T = e^{iuX_T}$, given the initial conditions $A(0) = B(0) = C(0) = D(0) = 0$.

[8]Use that

$$\begin{aligned} \frac{\partial G_t}{\partial v_t} &= G_t\left[B(t) + \frac{1}{2\sqrt{v_t}}D(t)\right] \\ \frac{\partial^2 G_t}{\partial v_t^2} &= G_t\left[B^2(t) + \frac{B(t)D(t)}{\sqrt{v_t}} + \frac{1}{4v_t}D^2(t) - \frac{1}{4(v_t)^{3/2}}D(t)\right]. \end{aligned}$$

5 Volatility-interest coupling

The third method discussed considers an interest rate process that is coupled[9] to the volatility, via

$$\mathrm{d}r_t = (\theta_t - ar_t)\mathrm{d}t + \sigma\sqrt{v_t}a_{3k}\mathrm{d}B_{k,t}.$$

The Feynman–Kac partial differential equation for the martingale $(e^{-R_t}G_t)$ then reads

$$\begin{aligned} r_tG_t = \frac{\partial G_t}{\partial t} + (r_t - \frac{1}{2}v_t)\frac{\partial G_t}{\partial X_t} + \kappa(\eta - v_t)\frac{\partial G_t}{\partial v_t} + (\theta_t - ar_t)\frac{\partial G_t}{\partial r_t} \\ + \frac{1}{2}v_t\frac{\partial^2 G_t}{\partial X_t^2} + \frac{1}{2}\lambda^2 v_t\frac{\partial^2 G_t}{\partial v_t^2} + \frac{1}{2}\sigma^2 v_t\frac{\partial^2 G_t}{\partial r_t^2} \\ + \lambda v_t\frac{\partial^2 G_t}{\partial X_t\partial v_t}\rho_{12} + \sigma v_t\frac{\partial^2 G_t}{\partial X_t\partial r_t}\rho_{13} + \lambda\sigma v_t\frac{\partial^2 G_t}{\partial v_t\partial r_t}\rho_{23}. \end{aligned}$$

Following Heston, we make a similar ansatz for the characteristic function:

$$G_t = e^{A(T-t)+B(T-t)v_t+C(T-t)r_t+iuX_t}.$$

Grouping together terms with v_t, respectively r_t, we get the following system of ordinary differential equations,

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= \kappa\eta B(t) + \theta_t C(t) \\ \frac{\mathrm{d}B}{\mathrm{d}t} &= b_0 + b_1 B(t) + \frac{1}{2}\lambda^2 B(t)^2 + \frac{1}{2}\sigma^2 C(t)^2 \\ &\qquad + \lambda\sigma\rho_{23}B(t)C(t) + iu\lambda\rho_{12}C(t) \\ \frac{\mathrm{d}C}{\mathrm{d}t} &= (iu - 1) + aC(t) \end{aligned}$$

where $b_0 = -\frac{1}{2}iu(1 - iu)$, and $b_1 = iu\sigma\rho_{13} - \kappa$.

The initial conditions are $A(0) = B(0) = C(0) = 0$, and the last equation has solution:

$$C(t) = \frac{1 - iu}{a}(e^{-at} - 1).$$

The second equation is a Riccati equation of form

$$\frac{\mathrm{d}B(t)}{\mathrm{d}t} = \frac{1}{2}\lambda^2 B(t)^2 + g(t)B(t) + h(t)$$

with coefficient functions

$$\begin{aligned} g(t) &= g_0 + g_1 e^{-at} \\ h(t) &= h_0 + h_1 e^{-at} + h_2 e^{-2at} \end{aligned}$$

[9] The form of this coupling is only motivated by the mathematical structure. In fact, whether this coupling is of any value in the modelling of real-world finance, is quite unclear, though one might expect it not to be.

where, setting $q = (1 - iu)$,

$$g_0 = iu\sigma\rho_{13} - \kappa - \lambda\sigma\frac{q}{a}\rho_{23}, \qquad h_0 = -\frac{1}{2}iuq + \frac{q^2\sigma^2}{2a^2} - iu\lambda\frac{q}{a}\rho_{12}$$

$$g_1 = \lambda\sigma\frac{q}{a}\rho_{23}, \qquad h_1 = iu\lambda\frac{q}{a}\rho_{12} - \frac{q^2\sigma^2}{a^2}$$

$$h_2 = \frac{q^2\sigma^2}{2a^2}.$$

Although the quadratic term $B(t)^2$ makes it impossible to split this equation into real and imaginary parts, there exists[10] an analytical solution of this equation in terms of Whittaker functions [1], such that it can be evaluated efficiently with tabulated values. Yet the equation for $A(t)$ makes it necessary to solve the whole system numerically. Still, this is more efficient than integration of the partial differential equation or direct (Monte-Carlo) simulation, and makes this approach also interesting.

6 Discussion

In this short note we have discussed three different ways of obtaining efficient solutions to extensions of the Heston model. Unfortunately, the page limitation in this contribution does not allow for numerical experiments with these methods.

A The method of Scott

Write

$$\begin{aligned}\ln S_t &= \int_0^t r_s \mathrm{d}s + \int_0^t \sqrt{v_s}\left(\rho \mathrm{d}W_{2,s} + \sqrt{1-\rho^2}\mathrm{d}W'_{2,s}\right) - \frac{1}{2}\int_0^t v_s \mathrm{d}s \\ &= R_t + \left(\rho\int_0^t \sqrt{v_s}\mathrm{d}W_{2,s} - \frac{1}{2}\rho^2\int_0^t v_s\mathrm{d}s\right) \\ &\quad + \left(\sqrt{1-\rho^2}\int_0^t \sqrt{v_s}\mathrm{d}W'_{2,s} - \frac{1}{2}(1-\rho^2)\int_0^t v_s\mathrm{d}s\right) \\ &= R_t + \eta_t + \xi_t.\end{aligned}$$

Since v_s develops independently from $\mathrm{d}W'_{2,s}$, we can calculate

$$\begin{aligned}\mathbb{E}[\xi_t \mid W_{2,t}] &= -\frac{1}{2}(1-\rho^2)V_t, \\ \mathrm{Var}[\xi_t \mid W_{2,t}] &= (1-\rho^2)V_t,\end{aligned}$$

where $V_t = \int_0^t v_s \mathrm{d}s$.

Furthermore, we now can use $\sqrt{v_t}\mathrm{d}W_{2,t} = \frac{1}{\lambda}(\mathrm{d}v_t - \kappa(\eta - v_t)\mathrm{d}t)$ to write

$$\eta_t = \frac{\rho}{\lambda}(v_t - v_0 - \kappa\eta t + \kappa V_t) - \frac{1}{2}\rho^2 V_t.$$

[10]The commercial software package Maple can be used to derive the analytical solution of this ODE.

Considering $\Psi(z) = \mathbb{E}\left[e^{-R_T + z \ln S_T}\right]$, we see that $\Psi(z) = \mathbb{E}\left[e^{(z-1)R_T}\right] \cdot \mathbb{E}\left[e^{z\xi_T + z\eta_T}\right]$. Now ξ_T, being an Itô integral, is normally distributed. Therefore $e^{z\xi_T}$ has a log-normal distribution, such that

$$\mathbb{E}[e^{z\xi_T} \mid W_{2,t}] = e^{(z-1)z\frac{1}{2}(1-\rho^2)V_t} \qquad \text{(conditional on } W_{2,t}\text{)}$$

and we arrive at the formula given in the text, Eq. (3.2).

References

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A semi closed-form analytic pricing formula for call options in a hybrid Heston–Hull–White model

Karel in 't Hout[*] *Joris Bierkens*[†] *Antoine P.C. van der Ploeg*[‡]
Jos in 't Panhuis[§]

1 Introduction

We consider the valuation of European call options under the general Heston–Hull–White asset pricing model. The model constitutes an extension of the well-known Black–Scholes model [3] where the volatility and the interest rate both evolve randomly over time. The process for the variance v_t has been proposed by Heston [5]. The process for the interest rate r_t was formulated by Hull and White [6] and forms a generalization of the Vasicek model [8]. In this contribution we assume that the process $W_{3,t}$ is independent from $W_{1,t}$ and $W_{2,t}$. The two Brownian motions $W_{1,t}$, $W_{2,t}$ are allowed to be correlated; their correlation is denoted by $\rho \in [-1, 1]$.

The purpose of this note is to derive an analytic pricing formula in semi closed-form for European call options under the Heston–Hull–White asset pricing model. The availability of such a pricing formula is particularly useful in a calibration procedure. In practice, option pricing models are calibrated to a large number of market-observed call option prices. It is important that such a parameter estimation procedure is fast. Therefore a (near) closed-form call option pricing formula is very desirable.

Our analysis in this note follows the lines of Heston [5]. The formula that we obtain forms a direct extension of Heston's pricing formula for call options, which can quickly be evaluated.

2 A semi closed-form analytic formula for call option prices

Let $C(t, s, v, r)$ denote the price of a European call option at time $t \in [0, T]$ given that at this time the asset price equals s, its variance equals v and the interest rate equals r.

From standard no-arbitrage arguments it follows that C satisfies the parabolic partial differential equation (PDE)

$$\begin{aligned} 0 \quad = \quad & \frac{\partial C}{\partial t} + \tfrac{1}{2}s^2 v \frac{\partial^2 C}{\partial s^2} + \tfrac{1}{2}\lambda^2 v \frac{\partial^2 C}{\partial v^2} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial r^2} + \rho\lambda s v \frac{\partial^2 C}{\partial s \partial v} \\ & + rs\frac{\partial C}{\partial s} + \kappa(\eta - v)\frac{\partial C}{\partial v} + (\theta(t) - ar)\frac{\partial C}{\partial r} - rC, \end{aligned} \tag{2.1}$$

[*]Universiteit Antwerpen, Belgium, karel.inthout@ua.ac.be
[†]Universiteit Leiden, joris@math.leidenuniv.nl
[‡]ING Corporate Market Risk Management/Quants, Amsterdam, Antoine.van.der.Ploeg@ingbank.com
[§]Technische Universiteit Eindhoven, j.c.h.w.panhuis@tue.nl
[*]Other participants: Claude Archer, Chun Dong Chau, Zhengwei Han, Remco van der Hofstad, David Kun

for $0 \le t < T$, $s > 0$, $v > 0$, $-\infty < r < \infty$. This PDE can be viewed as a time-dependent advection–diffusion–reaction equation on an unbounded, three-dimensional spatial domain. The payoff of a call option yields the terminal condition

$$C(T, s, v, r) = \max(0, s - K), \tag{2.2}$$

where $K > 0$ is the strike price of the call option. Further, a boundary condition at $s = 0$ holds,

$$C(t, 0, v, r) = 0 \quad (0 \le t < T). \tag{2.3}$$

We note that at $v = 0$ no condition is specified.

It is convenient to first apply a change of variables. Define

$$\hat{C}(t, x, v, r) = C(t, e^x, v, r). \tag{2.4}$$

Then $\hat{C}$ satisfies the PDE

$$\begin{aligned} 0 \quad = \quad & \frac{\partial \hat{C}}{\partial t} + \tfrac{1}{2} v \frac{\partial^2 \hat{C}}{\partial x^2} + \tfrac{1}{2}\lambda^2 v \frac{\partial^2 \hat{C}}{\partial v^2} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 \hat{C}}{\partial r^2} + \rho\lambda v \frac{\partial^2 \hat{C}}{\partial x \partial v} \\ & + (r - \tfrac{1}{2} v) \frac{\partial \hat{C}}{\partial x} + \kappa(\eta - v) \frac{\partial \hat{C}}{\partial v} + (\theta(t) - ar) \frac{\partial \hat{C}}{\partial r} - r\hat{C} \end{aligned} \tag{2.5}$$

for $0 \le t < T$ on the spatial domain $(x, v, r) \in \mathbb{R} \times (0, \infty) \times \mathbb{R}$ with terminal condition

$$\hat{C}(T, x, v, r) = \max(0, e^x - K). \tag{2.6}$$

As in [5], we guess a solution of the form similar to the Black–Scholes formula:

$$\hat{C}(t, x, v, r) = e^x P_1(t, x, v, r) - KB(t, r) P_2(t, x, v, r). \tag{2.7}$$

Here $B(t, r)$ denotes the time-t value of a zero-coupon bond that pays off 1 at maturity, given that at time t the short rate equals r. It satisfies the PDE

$$0 = \frac{\partial B}{\partial t} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 B}{\partial r^2} + (\theta(t) - ar)\frac{\partial B}{\partial r} - rB \tag{2.8}$$

for $0 \le t < T$, $r \in \mathbb{R}$ and a semi closed-form solution is given by

$$B(t, r) \quad = \quad e^{b(t,r)}, \tag{2.9a}$$

$$\begin{aligned} b(t, r) \quad = \quad & -\frac{r}{a}\left(1 - e^{-a(T-t)}\right) - \frac{1}{a}\int_t^T \theta(s)\left(1 - e^{-a(T-s)}\right) ds \\ & + \frac{\sigma^2}{2a^2}\left(T - t + \frac{2}{a} e^{-a(T-t)} - \frac{1}{2a} e^{-2a(T-t)} - \frac{3}{2a}\right). \end{aligned} \tag{2.9b}$$

By linearity, the guess (2.7) satisfies the PDE (2.5) if its two constituent terms satisfy (2.5). As such, P_1 satisfies the PDE

$$\begin{aligned} 0 \quad = \quad & \frac{\partial P_1}{\partial t} + \tfrac{1}{2} v \frac{\partial^2 P_1}{\partial x^2} + \tfrac{1}{2}\lambda^2 v \frac{\partial^2 P_1}{\partial v^2} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 P_1}{\partial r^2} + \rho\lambda v \frac{\partial^2 P_1}{\partial x \partial v} + \\ & (r + \tfrac{1}{2} v) \frac{\partial P_1}{\partial x} + [\kappa(\eta - v) + \rho\lambda v] \frac{\partial P_1}{\partial v} + (\theta(t) - ar) \frac{\partial P_1}{\partial r}, \end{aligned} \tag{2.10}$$

and by invoking (2.8), P_2 satisfies

$$\begin{aligned} 0 \quad = \quad & \frac{\partial P_2}{\partial t} + \tfrac{1}{2} v \frac{\partial^2 P_2}{\partial x^2} + \tfrac{1}{2}\lambda^2 v \frac{\partial^2 P_2}{\partial v^2} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 P_2}{\partial r^2} + \rho\lambda v \frac{\partial^2 P_2}{\partial x \partial v} + \\ & (r - \tfrac{1}{2} v) \frac{\partial P_2}{\partial x} + \kappa(\eta - v) \frac{\partial P_2}{\partial v} + \left[\theta(t) - ar + \sigma^2 \frac{\partial b}{\partial r}\right] \frac{\partial P_2}{\partial r} . \end{aligned} \tag{2.11}$$

Further, (2.6) yields for the PDEs (2.10), (2.11) the terminal conditions

$$P_j(T, x, v, r) = 1 \;\; (x > \ln K) \;\; , \;\; P_j(T, x, v, r) = 0 \;\; (x < \ln K) \tag{2.12}$$

for $j = 1, 2$, respectively.

From the undiscounted, multidimensional version of the Feynman–Kac Theorem (cf. [7]) it follows that the solutions P_1, P_2 to (2.10), (2.11) with (2.12) can be written as expectations of the indicator function corresponding to (2.12), and thus can be regarded as probabilities[1]. We next derive semi closed-form formulas for P_1 and P_2 by solving for their characteristic functions. From these characteristic functions the probabilities P_1, P_2 can be retrieved with the inversion theorem (cf. [4, 5]):

$$P_j(t, x, v, r) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{e^{-iu \ln K} f_j(t, x, v, r; u)}{iu}\right] du \quad \text{for } j = 1, 2 \tag{2.13}$$

where $i^2 = -1$.

The Feynman–Kac theorem directly yields that the functions f_1, f_2 satisfy the same PDEs (2.10), (2.11), respectively, but with the terminal condition

$$f_j(T, x, v, r; u) = e^{iux}. \tag{2.14}$$

For f_1 we guess a solution of the form (cf. [5])

$$f_1(t, x, v, r; u) = \exp[F_1(t; u) + G_1(t; u)v + H_1(t; u)r + iux]. \tag{2.15}$$

Substituting this into the PDE (2.10), it follows by perusal of the coefficients of v, r and 1 that (2.15) is a solution if the functions F_1, G_1, H_1 satisfy the system of ordinary differential equations (ODEs)

$$F_1'(t) + \kappa\eta G_1(t) + \theta(t) H_1(t) + \tfrac{1}{2}\sigma^2 H_1(t)^2 = 0\,, \tag{2.16a}$$

$$G_1'(t) + \tfrac{1}{2} ui - \tfrac{1}{2} u^2 + (\rho\lambda ui + \rho\lambda - \kappa) G_1(t) + \tfrac{1}{2}\lambda^2 G_1(t)^2 = 0\,, \tag{2.16b}$$

$$H_1'(t) + ui - aH_1(t) = 0\,, \tag{2.16c}$$

with the terminal condition $F_1(T) = G_1(T) = H_1(T) = 0$.

For f_2 we guess a solution of the form (cf. [2, 5])

$$f_2(t, x, v, r; u) = \exp[F_2(t; u) + G_2(t; u)v + H_2(t; u)r + iux - b(t, r)]. \tag{2.17}$$

Substituting this into the PDE (2.11) and using (2.8),(2.9), it follows analogously as above that (2.17) is a solution if the functions F_2, G_2, H_2 satisfy the system of ODEs

$$F_2'(t) + \kappa\eta G_2(t) + \theta(t) H_2(t) + \tfrac{1}{2}\sigma^2 H_2(t)^2 = 0\,, \tag{2.18a}$$

$$G_2'(t) - \tfrac{1}{2} ui - \tfrac{1}{2} u^2 + (\rho\lambda ui - \kappa) G_2(t) + \tfrac{1}{2}\lambda^2 G_2(t)^2 = 0\,, \tag{2.18b}$$

$$H_2'(t) + ui - aH_2(t) - 1 = 0\,, \tag{2.18c}$$

[1] We omit the details, which are completely analogous to those explained in [5].

with the terminal condition $F_2(T) = G_2(T) = H_2(T) = 0$.

The equations (2.16c), (2.18c) are easy to solve. Let $\delta_1 = 0$, $\delta_2 = 1$. Then

$$H_j(t; u) = \frac{ui - \delta_j}{a}\left(1 - e^{-a(T-t)}\right) \qquad \text{for } j = 1, 2. \tag{2.19}$$

The equations (2.16b), (2.18b) are identical[2] to the first line of equation (A7) in [5] and closed-form solutions were obtained in loc. cit. For completeness, we include these formulas here. Let

$$\alpha = \kappa\eta \ , \ \beta_1 = \kappa - \rho\lambda \ , \ \beta_2 = \kappa \ , \ \gamma_1 = \frac{1}{2} \ , \ \gamma_2 = -\frac{1}{2}$$

and for $j = 1, 2$

$$d_j = \sqrt{(\beta_j - \rho\lambda ui)^2 - \lambda^2(2\gamma_j ui - u^2)} \ , \ g_j = \frac{\beta_j - \rho\lambda ui + d_j}{\beta_j - \rho\lambda ui - d_j} \, .$$

Then the solutions to (2.16b), (2.18b) are given by

$$G_j(t; u) = \frac{\beta_j - \rho\lambda ui + d_j}{\lambda^2}\left[\frac{1 - e^{d_j(T-t)}}{1 - g_j e^{d_j(T-t)}}\right] \qquad \text{for } j = 1, 2. \tag{2.20}$$

The equations (2.16a), (2.18a) can finally be solved by integration. Using the result from [5] for the integral of G_j, it follows that

$$\begin{aligned} F_j(t; u) \ = \ & \frac{\alpha}{\lambda^2}\left\{(\beta_j - \rho\lambda ui + d_j)(T - t) - 2\ln\left[\frac{1 - g_j e^{d_j(T-t)}}{1 - g_j}\right]\right\} \\ & + \frac{ui - \delta_j}{a}\int_t^T \theta(s)\left(1 - e^{-a(T-s)}\right) ds \\ & + \frac{\sigma^2}{2}\left(\frac{ui - \delta_j}{a}\right)^2\left(T - t + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right) \end{aligned} \tag{2.21}$$

for $j = 1, 2$. Of course, for many functions θ the integral in (2.21) may be explicitly computed.

The formulas (2.4), (2.7), (2.9), (2.13), (2.15), (2.17), (2.19), (2.20), (2.21) together constitute the semi closed-form pricing formula for European call options under the hybrid asset pricing model. This pricing formula is easily seen to be a proper extension of Heston's formula, upon considering $\theta(t) \equiv ar_0$ and $\sigma = 0$.

If the integrals in (2.9b), (2.21) involving $\theta(s)$ can be explicitly computed, the pricing formula consists of two single integrals over u, see (2.13). Otherwise, one has an additional single integral over s,

$$\int_t^T \theta(s)\left(1 - e^{-a(T-s)}\right) ds \, .$$

Note the useful property that the latter integral does not depend on u. In all cases, the pricing formula can be quickly approximated to any accuracy with a suitable numerical integration method. For a discussion of some computational issues relevant to the pricing formula, we refer to the paper [1] on the Heston formula.

[2] With the proper change of notation and removing a typo in [5].

Finally, we remark that two issues are not addressed in this note, namely whether the solution obtained above is unique and whether it satisfies the condition (2.3). These two issues are left for future research. We note that it is plausible that the probability $P_2(t,x,v,r)$ in (2.7) vanishes as $x \to -\infty$, and therefore that (2.3) holds. But, this requires a careful analysis of course.

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Characteristic function of the hybrid Heston–Hull–White model

Fang Fang[*] *Bas Janssens*[†]

In our contribution the goal is to find the analytic solution of the characteristic function (ch.f.) of x_T, given the initial data under the hybrid Heston–Hull–White model. That is, we want to find a closed form expression for

$$\Phi(\omega; x_0, v_0, r_0) := \mathbb{E}(\exp(i\omega x_T)|x_0, v_0, r_0).$$

A first observation on the model is the following: If x_t satisfies

$$dx_t = (r_t - \tfrac{1}{2}v_t)dt + \sqrt{v_t}d\tilde{W}_{1,t}\,,$$

then $S_t = \exp(x_t)$ satisfies the Heston–Hull–White model, as can be seen by applying Itô's lemma. This paper has a twofold aim:

- Solve the problem under the assumption $\rho_{13} = \rho_{23} = 0$.
- Solve the problem under the assumption $\rho_{23} = 0$, and under the additional assumption that $\kappa\eta = \lambda^2/4$, in which case $\sqrt{v_t}$ is governed by an Ornstein–Uhlenbeck process.

It is organized as follows: in section 1, we decompose the three correlated Wiener processes into three independent ones, and establish some notation. In section 2, we eliminate two noises by exploiting the Gaussianity of the r_t-distribution, as well as the fact that x_t does not occur on the r.h.s. of the equations. In section 3, we obtain the ch.f. of x_T in the aforementioned two cases.

1 Reformulating the Model

With the assumption that $\rho_{23} = 0$, we can write $\tilde{W}_{i,t}$, $i = 1, 2, 3$, as a sum of *independent* processes $W_{i,t}$:

$$\begin{aligned}\tilde{W}_{3,t} &= W_{3,t}\\ \tilde{W}_{2,t} &= W_{2,t}\\ \tilde{W}_{1,t} &= \alpha_1 W_{1,t} + \alpha_2 W_{2,t} + \alpha_3 W_{3,t},\end{aligned}$$

where $\alpha_2 = \rho_{12}$, $\alpha_3 = \rho_{13}$, and $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$. Thus the model is reformulated as

$$dx_t = (r_t - \tfrac{1}{2}v_t)dt + \alpha_1\sqrt{v_t}dW_{1,t} + \alpha_2\sqrt{v_t}dW_{2,t} + \alpha_3\sqrt{v_t}dW_{3,t} \quad (1.1)$$

$$dv_t = \kappa(\eta - v_t)dt + \lambda\sqrt{v_t}dW_{2,t} \quad (1.2)$$

$$dr_t = (\theta(t) - ar_t)dt + \sigma dW_{3,t}\,. \quad (1.3)$$

[*]Technische Universiteit Delft, f.fang@ewi.tudelft.nl
[†]Universiteit Utrecht, janssens@math.uu.nl

Equation (1.2) gives $\sqrt{v_t} dW_{2,t} = (dv_t - \kappa(\eta - v_t)dt)/\lambda$. Insert it into (1.1) to obtain

$$dx_t = r_t dt + \left(\frac{\alpha_2 \kappa}{\lambda} - \tfrac{1}{2}\right) v_t dt - \frac{\alpha_2 \kappa \eta}{\lambda} dt + \frac{\alpha_2}{\lambda} dv_t + \alpha_1 \sqrt{v_t} dW_{1,t} + \alpha_3 \sqrt{v_t} dW_{3,t} \tag{1.4}$$

(cf. [1].) We introduce the notation

$$R_t := \int_0^t r_s ds \quad \text{and} \quad V_t := \int_0^t v_s ds \,.$$

Equation (1.4) is then integrated to

$$x_T - x_0 = R_T + \left(\frac{\alpha_2 \kappa}{\lambda} - \tfrac{1}{2}\right) V_T - \frac{\alpha_2 \kappa \eta}{\lambda} T + \frac{\alpha_2}{\lambda}(v_T - v_0) + \alpha_1 \int_0^T \sqrt{v_t} dW_{1,t} + \alpha_3 \int_0^T \sqrt{v_t} dW_{3,t}.$$

From now on, unless otherwise specified, all expectations are understood to be conditioned on x_0, r_0 and v_0, i.e.

$$\mathbb{E}(X) := \mathbb{E}(X|x_0, v_0, r_0).$$

Using the tower property of conditional expectations, we have

$$\begin{aligned}\Phi(\omega; x_0, v_0, r_0) &= \mathbb{E}\left\{\mathbb{E}\left[e^{i\omega(x_T - x_0)}|R_T, \{v_s; s \in [0,T]\}\right]\right\} \\ &= \mathbb{E}\Big\{ \exp\left(i\omega[R_T + \left(\frac{\alpha_2\kappa}{\lambda} - \tfrac{1}{2}\right) V_T - \frac{\alpha_2 \kappa \eta}{\lambda} T + \frac{\alpha_2}{\lambda}(v_T - v_0)]\right) \\ &\times \mathbb{E}\left[\exp\left(i\omega[\alpha_1 \int_0^T \sqrt{v_t} dW_{1,t} + \alpha_3 \int_0^T \sqrt{v_t} dW_{3,t}]\right)|R_T, \{v_s; s \in [0,T]\}\right]\Big\}.\end{aligned} \tag{1.5}$$

Note that v_t and R_T are only driven by their own noises, but $\Phi(\omega)$ is still driven by all three Wiener processes.

2 Elimination of Two Noises

As the title suggests, two driving noises will be eliminated in this section.

2.1 Distribution of R_T

The dynamics of the interest rate r_t can be rewritten as follows:

$$\begin{aligned} dr_t &= (\theta(t) - a r_t) dt + \sigma dW_{3,t} \\ d(e^{at} r_t) &= e^{at}\theta(t) dt + e^{at} \sigma dW_{3,t} \\ r_\tau &= e^{-a\tau} r_0 + \int_0^\tau \theta(s) e^{a(s-\tau)} ds + \sigma \int_0^\tau e^{a(s-\tau)} dW_{3,s}. \end{aligned}$$

Thus for $R_T := \int_0^T r_\tau d\tau$, we have nested integrals. Fubini's theorem yields

$$\int_0^T \left(\int_0^\tau \theta(s) e^{a(s-\tau)} ds \right) d\tau = \frac{1}{a} \int_0^T \theta(s) \left(1 - e^{a(s-T)}\right) ds.$$

For the stochastic part, we have

$$\int_0^T \left(\int_0^\tau e^{a(s-\tau)} dW_{3,s} \right) d\tau = \frac{1}{a} \int_0^T \left(1 - e^{a(s-T)} \right) dW_{3,s}.$$

Patching these together, we obtain

$$R_T = F_{r_0,a,\theta}(T) + \frac{\sigma}{a} \int_0^T \left(1 - e^{a(s-T)} \right) dW_{3,s}, \tag{2.1}$$

with

$$F_{r_0,a,\theta}(T) := \frac{r_0}{a} \left(1 - e^{-aT} \right) + \frac{1}{a} \int_0^T \theta(s) \left(1 - e^{a(s-T)} \right) ds\,.$$

Since the Itô integral in (2.1) is a weighted Wiener process, R_T has a Gaussian distribution with mean $F(T)$ and variance

$$\mathrm{Var}(T) := \frac{\sigma^2}{a^2} \int_0^T \left(1 - e^{a(s-T)} \right)^2 ds = \frac{\sigma^2}{a^2} (T - \frac{2}{a}(1 - e^{-aT}) + \frac{1}{2a}(1 - e^{-2aT})).$$

2.2 The Correlation

Recall the expression for Φ in (1.5). Let us first focus on the inner expectation, which is conditioned on R_T and the complete path $\{v_s\}_{0 \le s \le T}$.

We fix a function $w : [0, T] \to \mathbb{R}^+$ such that $v_s = w(s)$ and introduce the notation $W_i(f) := \int_0^T f(s) dW_{i,s}$.

Since there is no restriction on x_s, the process $W_{1,t}$ is still a Wiener process. With fixed $v_s = w(s)$, the random variable $W_{1,t}(\sqrt{w})$ is just a weighted Brownian motion. For $W_{3,t}$ however, there *is* a restriction, since we have fixed R_T. When we define $g(s) := (1 - e^{a(s-T)})$, we have fixed

$$W_{3,t}(g) = \frac{a}{\sigma}(R_T - F_{r_0,a,\theta}(T))\,.$$

Apart from the fixed x_0 , r_0 and v_s, this is the *only* relevant restraint. And since $W_{3,t}(f)$ is independent from $W_{3,t}(g)$ if $f \perp g$ we simply decompose

$$W_{3,t}(\sqrt{w}) = W_{3,t}(\sqrt{w}_{\|}) + W_{3,t}(\sqrt{w}_{\perp})$$

with

$$\sqrt{w}_{\|} = \frac{\langle \sqrt{w}, g \rangle}{\langle g, g \rangle} g \quad \text{and} \quad \sqrt{w}_{\perp} = \sqrt{w} - \frac{\langle \sqrt{w}, g \rangle}{\langle g, g \rangle} g.$$

Thus, for fixed v_s and R_T, we know that $W_{3,t}(\sqrt{w})$ is Gaussianly distributed with mean

$$\frac{a}{\sigma}(R_T - F_{r_0,a,\theta}(T)) \times \frac{\langle \sqrt{w}, g \rangle}{\langle g, g \rangle}$$

and variance

$$\langle \sqrt{w}_{\perp}, \sqrt{w}_{\perp} \rangle = \langle \sqrt{w}, \sqrt{w} \rangle - \frac{\langle \sqrt{w}, g \rangle^2}{\langle g, g \rangle}\,,$$

where $\langle g, g \rangle = T - \frac{2}{a}(1 - e^{-aT}) + \frac{1}{2a}(1 - e^{-2aT})$. If we define

$$\mu_T := \int_0^T (1 - e^{a(s-T)}) \sqrt{v_s} ds\,,$$

we have

$$W_{3,t}(\sqrt{w}) \sim N\left(\frac{a\mu_T}{\sigma\langle g,g\rangle}(R_T - F(T)),\, V_T - \frac{\mu_T^2}{\langle g,g\rangle}\right).$$

Recall that $W_{1,t}$, $W_{2,t}$ and $W_{3,t}$ are independent Wiener processes. The process R_t is only driven by $W_{3,t}$, and v_t only by $W_{2,t}$. For fixed R_T and $v_s = w(s)$, we therefore know that $W_{1,t}(\sqrt{w})$ and $W_{3,t}(\sqrt{w})$ are independent Gaussians. The conditional characteristic function (CCF) of their sum,

$$\phi(\omega; R_T, v_s) := \mathbb{E}\Big[\exp\Big(i\omega(\alpha_1 W_{1,t}(\sqrt{w}) + \alpha_3 W_{3,t}(\sqrt{w}))\Big)|R_T, \{v_s; s \in [0,T]\}\Big],$$

is therefore the product of the individual CCF's for $\alpha_1 W_{1,t}(\sqrt{w})$ and $\alpha_3 W_{3,t}(\sqrt{w})$. These we know; the characteristic function of a Gaussian with mean μ and variance u is

$$f_{\mu,u}(\omega) = \exp(i\mu\omega - \frac{u}{2}\omega^2).$$

Adding the means and variances of the two independent Gaussians, we write

$$\phi(\omega; R_T, v_s) = f_{\alpha_3\frac{a\mu_T}{\sigma\langle g,g\rangle}(R_T-F(T)),(\alpha_1^2+\alpha_3^2)V_T-\alpha_3^2\frac{\mu_T^2}{\langle g,g\rangle}}(\omega)$$

which only depends on R_T, μ_T and V_T.

Returning to (1.5), we have

$$\begin{aligned}\Phi(\omega; x_0, v_0, r_0) &= \mathbb{E}\Big[\exp(i\omega\Big(R_T + \Big(\frac{\alpha_2\kappa}{\lambda} - \tfrac{1}{2}\Big)V_T - \frac{\alpha_2\kappa\eta}{\lambda}T + \frac{\alpha_2}{\lambda}(v_T - v_0)\Big))\\ &\times\exp(i\omega\Big(\alpha_3\frac{a\mu_T}{\sigma\langle g,g\rangle}[R_T - F(T)]\Big))\\ &\times\exp(-\tfrac{1}{2}\omega^2\Big((\alpha_1^2+\alpha_3^2)V_T - \alpha_3^2\frac{\mu_T^2}{\langle g,g\rangle}\Big))\Big].\end{aligned} \tag{2.2}$$

We now use the tower rule to get an inner expectation conditioned on μ_T , V_T and v_T. That is:

$$\Phi(\omega; x_0, v_0, r_0) = \mathbb{E}(\ldots) = \mathbb{E}(\mathbb{E}(\ldots|\mu_T, V_T, v_T)).$$

Recall that R_T is independent of V_T, v_T, or μ_T. So what remains as the inner expectation is $\mathbb{E}(e^{i\omega c(R_T-F(T))}|\mu_T, V_T, v_T)$, with $c = (1 + \alpha_3\frac{a}{\sigma\langle g,g\rangle}\mu_T)$. Since $R_T - F(T)$ is $N(0, \frac{\sigma^2}{a^2}\langle g,g\rangle)$-distributed, we have

$$\mathbb{E}(\exp(i\omega c(R_T - F(T)))|\mu_T, V_T, v_T) = f_{0,\frac{\sigma^2}{a^2}\langle g,g\rangle}(c\omega) = \exp(-\frac{\sigma^2}{2a^2}\langle g,g\rangle c^2\omega^2).$$

In writing this out, the term $-\frac{1}{2}\frac{\alpha_3^2}{\langle g,g\rangle}\mu_T^2\omega^2$ mysteriously vanishes;

$$\begin{aligned}\Phi(\omega; x_0, v_0, r_0) &= \exp(i\omega F(T) - i\omega\frac{\alpha_2\kappa\eta}{\lambda}T - \tfrac{1}{2}\omega^2\frac{\sigma^2}{a^2}\langle g,g\rangle)\\ &\times\mathbb{E}\Big[e^{i\omega\frac{\alpha_2}{\lambda}(v_T-v_0)} \times e^{i\omega[(\frac{\alpha_2\kappa}{\lambda}-\frac{1}{2})+\frac{1}{2}i\omega(\alpha_1^2+\alpha_3^2)]V_T} \times e^{i\omega[i\omega\alpha_3\frac{\sigma}{a}]\mu_T}\Big].\end{aligned} \tag{2.3}$$

We simplify the expression by introducing the notations

$$C_0 := e^{i\omega F(T)-i\omega\frac{\alpha_2\kappa\eta}{\lambda}T-\frac{1}{2}\omega^2\frac{\sigma^2}{a^2}\langle g,g\rangle}, \quad C_1 := \frac{\alpha_2}{\lambda},$$

$$C_2 := \frac{\alpha_2\kappa}{\lambda} - \frac{1}{2} + \frac{1}{2}i\omega(\alpha_1^2 + \alpha_3^2), \quad C_3 := i\omega\alpha_3\sigma/a,$$

and

$$Z_T - Z_0 := C_1(v_T - v_0) + C_2 V_T + C_3\mu_T. \tag{2.4}$$

Thus we have

$$\Phi(\omega; x_0, v_0, r_0) = C_0\mathbb{E}[\exp(i\omega(Z_T - Z_0))]. \tag{2.5}$$

Therefore, finding the ch.f. of x_T at ω is equivalent to finding the ch.f. of $Z_T - Z_0$ at ω (where Z_t still depends on ω through the constants $C_i(\omega)$).

Integrating dv_t over $[0, T]$ gives

$$v_T - v_0 = \int_0^T \kappa(\eta - v_s)ds + \int_0^T \lambda\sqrt{v_s}dW_{2,s}.$$

Substituting this into (2.4) yields

$$Z_T - Z_0 = \int_0^T \left[C_1\kappa(\eta - v_s) + C_2 v_s + C_3 g(s)\sqrt{v_s}\right]ds + C_1\int_0^T \lambda\sqrt{v_s}dW_{2,s}. \tag{2.6}$$

Equivalently, the dynamics of Z_t read

$$dZ_t = \left[C_1\kappa\eta + (C_2 - C_1\kappa)v_t + C_3 g(t)\sqrt{v_t}\right]dt + C_1\lambda\sqrt{v_t}dW_{2,t}. \tag{2.7}$$

We have a small subtlety here. In principle, $g(s, T)$ depends both on s and on T. In the dynamics of Z_t, this would give rise to terms involving $\partial g/\partial T$. We circumvent this problem by defining, for each fixed T, a process $t \mapsto \hat{Z}(T)_t$ which is defined according to equation (2.7), in which $g(s, T)$ has *fixed* T. Then $Z_T = \hat{Z}_T(T)$. *From now on, we work with equation* (2.7), *omitting the hats.*

All in all, we are left with Z_t, which is driven by a single noise, $W_{2,t}$.

3 Analytic Solution

We denote the ch.f. of Z_T conditioned on $\mathcal{F}_t$ by

$$\Psi_t(\omega; \mathcal{F}_t) := \mathbb{E}\left[\exp(i\omega Z_T)\,|\mathcal{F}_t\right].$$

By definition, Ψ_t is a martingale: $\mathbb{E}\left[d\Psi_t|\mathcal{F}_t\right] = 0$. It is clear that Ψ_t depends only on $Z_t, v_t, \sqrt{v_t}, t$. Therefore, Itô's lemma yields, setting $\tau = T - t$:

$$\begin{aligned} d\Psi(\omega; Z_t, v_t, \sqrt{v_t}, \tau) = & -\frac{\partial\Psi}{\partial\tau}dt + \frac{\partial\Psi}{\partial Z_t}dZ_t + \frac{\partial\Psi}{\partial v_t}dv_t + \frac{\partial\Psi}{\partial\sqrt{v_t}}d\sqrt{v_t} + \frac{1}{2}\frac{\partial^2\Psi}{\partial Z_t^2}(dZ_t)^2 \\ & + \frac{1}{2}\frac{\partial^2\Psi}{\partial v_t^2}(dv_t)^2 + \frac{1}{2}\frac{\partial^2\Psi}{\partial\sqrt{v_t}^2}(d\sqrt{v_t})^2 + \frac{\partial^2\Psi}{\partial Z_t\partial v_t}dZ_t dv_t \\ & + \frac{\partial^2\Psi}{\partial Z_t\partial\sqrt{v_t}}dZ_t d\sqrt{v_t} + \frac{\partial^2\Psi}{\partial v_t\partial\sqrt{v_t}}dv_t d\sqrt{v_t}. \end{aligned} \tag{3.1}$$

Given the dynamics of $\sqrt{v_t}$, this gives rise to a PDE for Ψ. We proceed with the two cases in which we can solve this.

3.1 Case 1: $\rho_{23} = 0$ and $\rho_{13} = 0$

This is essentially the Heston model. Indeed, it is immediately clear from equation (1.1) that $x_t = x_{H,t} + R_t - r_0 t$, where $x_{H,t}$ denotes the logarithmic price in the Heston model. Therefore,

$$\Phi(\omega; x_0, v_0, r_0) = \chi(\omega)\Phi_H(\omega; x_0, v_0, r_0),$$

with $\chi(\omega) = e^{i\omega F(T) - i\omega r_0 T - \frac{1}{2}\omega^2 \frac{\sigma^2}{a^2}\langle g,g\rangle}$ the characteristic function of $R_T - r_0 T$.

3.2 Case 2: $\rho_{23} = 0$ and $\kappa\eta = \lambda^2/4$

We will now solve a different set of equations:

$$\begin{aligned} dx_t &= (r_t - \tfrac{1}{2}v_t)dt + \Theta_t d\tilde{W}_{1,t} \quad &(3.2)\\ d\Theta_t &= -\beta\Theta_t dt + \delta dW_{2,t} \quad &(3.3)\\ dr_t &= (\theta(t) - ar_t)dt + \sigma W_{3,t} \quad &(3.4)\end{aligned}$$

The relevance is as follows: if we set $B_{2,t} := \int_0^t sn(\Theta_s)dW_{2,s}$, with $sn(x)$ the sign of x, then $t \mapsto B_{2,t}$ is again a Wiener process by Levy's Martingale characterization of Brownian motions. We then see that $v_t = \Theta_t^2$ satisfies (the 2^{nd} equation of) the hybrid Heston–Hull–White model for the particular case $\kappa\eta = \lambda^2/4$.

Indeed, $d\Theta_t^2 = (\delta^2 - 2\beta\Theta_t^2)dt + 2\delta\Theta dW_{2,t}$ and $dW_{2,t} = sn(\Theta_t)dB_{2,t}$, so that

$$dv_t = (\delta^2 - 2\beta v_t)dt + 2\delta\sqrt{v_t}dB_{2,t}.$$

(Recall that $\sqrt{v_t} = |\Theta_t| = sn(\Theta_t)\Theta_t$.) This requires $\lambda = 2\delta$, $\kappa = 2\beta$ and $\kappa\eta = \delta^2$, and thus $\kappa\eta = \lambda^2/4$. This subclass of the model, in which Θ_t is an Ornstein–Uhlenbeck process, was in fact Heston's way of justifying the more general model [2]. Notice that we have changed $\sqrt{v_t}$ into Θ in equation (3.2). This seems to be essential unless $\rho_{13} = 0$.

We change to a new measure, under which Θ_t is simply Brownian motion [3]. Define the Radon-Nikodym derivative as

$$M_\tau = \exp(-Y_\tau) := \exp\left(-\int_0^\tau \frac{-\beta\Theta_t}{\delta}dW_{2,t} - \tfrac{1}{2}\int_0^\tau \frac{\beta^2\Theta_t^2}{\delta^2}dt\right).$$

Substitute $\int_0^\tau \Theta_t dW_{2,t}$ by $\frac{1}{2\delta}\left[(v_\tau - v_0) + \int_0^\tau 2\beta\Theta_t^2 dt - \delta^2\tau\right]$, we have

$$Y_\tau = -\frac{\beta}{2\delta^2}(v_\tau - v_0) - \frac{\beta^2}{2\delta^2}V_\tau + \frac{\beta}{2}\tau.$$

By Girsanov's theorem, if $\mathbb{E}\left[\exp\left(\int_0^T \frac{\beta^2}{\delta^2}v_t dt\right)\right] < \infty$, then

$$\hat{B}_\tau := \int_0^\tau \frac{-\beta\Theta_t}{\delta}dt + W_{2,\tau}; \qquad \tau \le T$$

is a Brownian motion w.r.t. $d\mathbb{Q} = M_T d\mathbb{P}$, where $\mathbb{P}$ denotes the old measure. Moreover, in terms of $\hat{B}_t$, the process Θ_t has the representation of

$$d\Theta_t = \delta d\hat{B}_t.$$

Thus

$$dv_t = d\Theta_t^2 = 2\Theta_t d\Theta_t + d\Theta_t^2 = 2\delta\Theta_t d\hat{B}_t + \delta^2 dt.$$

The derivation of equation (2.7) goes through word for word in this new system, except that all $\sqrt{v}$'s are replaced by Θ's. (In particular, μ should be defined as $\mu_t = \int_0^t g(s)\Theta_s ds$.) We obtain

$$dZ_t = \left[C_1\kappa\eta + (C_2 - C_1\kappa)\Theta_t^2 + C_3 g(t)\Theta_t\right] dt + C_1\lambda\Theta_t dW_{2,t}. \tag{3.5}$$

For the expectation w.r.t. $\mathbb{P}$, one has

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\exp(i\omega(Z_T - Z_0))\right] &= \int_\Omega \exp\left(i\omega(Z_T - Z_0)\right) M_T^{-1} d\mathbb{Q} \\ &= \int_\Omega \exp\left(i\omega(Z_T - Z_0)\right)\exp(Y_T) d\mathbb{Q} := \mathbb{E}^{\mathbb{Q}}\left[\exp(i\omega(\tilde{Z}_T - \tilde{Z}_0))\right],\end{aligned}$$

with (we use v_t for Θ_t^2 and V_t for its integral)

$$\tilde{Z}_T - \tilde{Z}_0 = (C_1 + i\frac{\beta}{2\omega\delta^2})(v_T - v_0) + (C_2 + i\frac{\beta^2}{2\omega\delta^2})V_T + C_3\mu_T - i\frac{\beta}{2\omega}T,$$

and

$$d\tilde{Z}_t = \left[C_1\delta^2 + (C_2 + i\frac{\beta^2}{2\omega\delta^2})v_t + C_3 g(t)\Theta_t\right] dt + (2\delta C_1 + i\frac{\beta}{\omega\delta})\Theta_t d\hat{B}_t.$$

Let us then set $\tau := T - t$. Our ansatz for Ψ will be

$$\Psi(\tilde{Z}_t, v_t, \Theta_t, \tau) = \exp\left[C(\tau) + D(\tau)\Theta_t + E(\tau)v_t + i\omega\tilde{Z}_t\right], \tag{3.6}$$

with initial conditions

$$C(0) = 0, D(0) = 0 \text{ and } E(0) = 0.$$

For this we have

$$\begin{aligned}\frac{\partial\Psi}{\partial\tau}/\Psi &= \frac{\partial C}{\partial\tau} + \frac{\partial D}{\partial\tau}\Theta_t + \frac{\partial E}{\partial\tau}v_t, & \frac{\partial\Psi}{\partial\tilde{Z}_t}/\Psi &= i\omega, \\ \frac{\partial\Psi}{\partial v_t}/\Psi &= E, & \frac{\partial\Psi}{\partial\Theta_t}/\Psi &= D, \\ \frac{\partial^2\Psi}{\partial\tilde{Z}_t^2}/\Psi &= -\omega^2, & \frac{\partial^2\Psi}{\partial v_t^2}/\Psi &= E^2, \\ \frac{\partial^2\Psi}{\partial\Theta_t^2}/\Psi &= D^2, & \frac{\partial^2\Psi}{\partial v_t\partial\tilde{Z}_t}/\Psi &= i\omega E, \\ \frac{\partial^2\Psi}{\partial\Theta_t\partial\tilde{Z}_t}/\Psi &= i\omega D, & \frac{\partial^2\Psi}{\partial\Theta_t\partial v_t}/\Psi &= DE.\end{aligned}$$

Substituting these into (3.1), factoring out Ψ and remembering $\mathbb{E}(d\Psi) = 0$, we come to the following equation:

$$\begin{aligned}0 = &\left(-\frac{\partial C}{\partial\tau} + i\omega C_1\delta^2 + \delta^2 E + \tfrac{1}{2}\delta^2 D^2\right) \\ &+ v_t\left(-\frac{\partial E}{\partial\tau} + i\omega C_4 - \tfrac{1}{2}\omega^2 C_5^2 + 2\delta^2 E^2 + 2i\omega\delta C_5 E\right) \\ &+ \Theta_t\left(-\frac{\partial D}{\partial\tau} + i\omega C_3 g(t) + i\omega\delta C_5 D + 2\delta^2 ED\right),\end{aligned} \tag{3.7}$$

with $C_4 := C_2 + i\frac{\beta^2}{2\omega\delta^2}$ and $C_5 := 2\delta C_1 + i\frac{\beta}{\omega\delta}$. Since (3.7) has to hold for every v_t and Θ_t, we obtain three ODEs:

$$-\frac{\partial C}{\partial\tau} + i\omega C_1\delta^2 + \delta^2 E + \tfrac{1}{2}\delta^2 D^2 = 0 \tag{3.8}$$

$$-\frac{\partial E}{\partial\tau} + i\omega C_4 - \tfrac{1}{2}\omega^2 C_5^2 + 2\delta^2 E^2 + 2i\omega\delta C_5 E = 0 \tag{3.9}$$

$$-\frac{\partial D}{\partial\tau} + i\omega C_3 g(t) + i\omega\delta C_5 D + 2\delta^2 ED = 0\,. \tag{3.10}$$

With $\gamma := \delta\sqrt{-2i\omega C_4}$, we find

$$E(\tau) = e_+\frac{1-\exp[2\delta^2(e_+-e_-)\tau]}{1-\frac{e_+}{e_-}\exp[2\delta^2(e_+-e_-)\tau]}, \tag{3.11}$$

$$\begin{aligned} D(\tau) = {} & \frac{i\omega C_3 e^{\gamma\tau}}{\frac{e_+}{e_-}e^{2\gamma\tau}-1}\times\Big(\big(\frac{1}{\gamma}(e^{-\gamma\tau}-1)-\frac{1}{a+\gamma}(e^{-(a+\gamma)\tau}-1)\big) \\ & +\frac{e_+}{e_-}\big(\frac{1}{\gamma}(e^{\gamma\tau}-1)-\frac{1}{\gamma-a}(e^{(\gamma-a)\tau}-1)\big)\Big), \end{aligned} \tag{3.12}$$

$$\begin{aligned} C(\tau) = {} & (e_+ + i\omega C_1)\delta^2\tau - \frac{1}{2}\log(\tfrac{e_+}{e_-}e^{2\delta^2(e_+-e_-)\tau}-1) + \\ & \tfrac{1}{2}\int_0^\tau D^2(s)ds\,. \end{aligned} \tag{3.13}$$

We briefly sketch how to arrive at this. Reformulate (3.9) as

$$\frac{d}{d\tau}\left[\log(E-e_+)-\log(E-e_-)\right] = 2\delta^2(e_+-e_-),$$

with $e_\pm = \frac{-i\omega C_5\pm\sqrt{-2i\omega C_4}}{2\delta}$. This yields (3.11). For (3.12), we first solve the homogeneous equation

$$\frac{dD_0}{d\tau} = i\omega\delta C_5 D_0 + 2\delta^2 ED_0.$$

Explicitly,

$$D_0(\tau) = \exp(i\omega\delta C_5\tau + 2\delta^2\int_0^\tau E(s)ds),$$

and[1]

$$\int_0^\tau E(s)ds = e_+\tau - \frac{1}{2\delta^2}\log(\frac{e_+}{e_-}e^{2\delta^2(e_+-e_-)\tau}-1).$$

Thus

$$D_0(\tau) = \frac{\exp((2\delta^2 e_+ + i\omega\delta C_5)\tau)}{\frac{e_+}{e_-}\exp(2\delta^2(e_+-e_-)\tau)-1}\,.$$

From 'variation of constants', we see $D(\tau) = i\omega C_3 D_0(\tau)\int_0^\tau g(T-s)D_0^{-1}(s)ds$, with $g(T-s) = (1-e^{-as})$. The result of this laborious but simple integration is shown above. (One uses $2\delta^2(e_+-e_-) = 2\gamma$ and $2\delta^2 e_\pm + i\omega\delta C_5 = \pm\gamma$.) The result for $C(\tau)$ is obtained by integration, where we have left the term $\int_0^\tau D^2(s)ds$ intact.

Substituting the above into (3.6) and (2.5), we obtain the analytic solution for ch.f. of x_T given x_0, v_0 and r_0.

[1] In principle, the logarithm should be interpreted carefully, in the sense that the function should not jump at branch cuts. However, since it occurs inside an exponential eventually, this remark belongs in a footnote.

4 Conclusion

Towards a solution to the problem of finding the characteristic function of the Heston–Hull–White model, we have made the following observations:

- With ρ_{13} and ρ_{23} equal to zero, the problem is essentially equivalent to Heston's model, and can be solved.
- With $\rho_{23} = 0$ and $\kappa\eta = \lambda^2/4$, but with arbitrary ρ_{13}, the problem can also be solved. This is an extension of the model by Stein and Stein [4]. It is tractable because the process underlying the volatility is an Ornstein–Uhlenbeck process, and not a Bessel process as in the Heston model. This is exactly the special case used by Heston to motivate the general model.

With only $\rho_{23} = 0$, the problem seems to be less simple. Still, we have been able to eliminate two out of three driving noises, which may result in faster numerical simulation.

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