Measure under Pressure

Calibration of pressure measurement

Magdalena Caubergh[1], Jan Draisma[2], Geert-Jan Franx[3], Geertje Hek[4], Georg Prokert[2], Sjoerd Rienstra[2], Arie Verhoeven[2]

Abstract

Piston-cylinder assemblies are used to create a calculable pressure in a container, which can then be used for calibration of other instruments. For this purpose one needs to calculate the pressure in the container so accurately that both imperfections in the piston, and the leakage of fluid or gas through the small space between cylinder and piston have to be taken into account. Because of these effects, the piston behaves as if its area was slightly larger than it actually is. This slightly larger area is called the *effective area* of the piston-cylinder assembly, and its computation is the subject of this report.

We derive a formula for this effective area, which under some simplifications leads to the formula used by four European metrological institutes. The formula used by NMi is based on a further simplification. We conclude with some recommendations to NMi concerning which formula to use and how to compute the uncertainty in the results.

Keywords: effective area, piston-cylinder assemblies, pressure balance, thin film approximation.

1.1 Problem description

Six European metrological institutes have compared their respective methods of calculating the effective area of piston-cylinder assemblies, which are used for calibration of pressure measurements [6]. Among them was NMi (Nederlands Meetinstituut = Dutch metrological institute), whose method and results were quite different from those of the other five institutes. NMi asked the study group Mathematics with Industry: first, to explain the differences between the

1: Universiteit Hasselt, 2: Technische Universiteit Eindhoven, 3: Vrije Universiteit Amsterdam, 4: Universiteit van Amsterdam

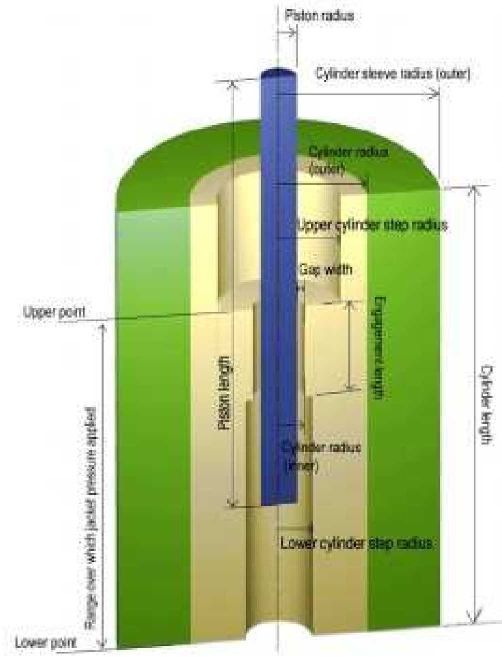

Figure 1.1: Basic geometry of the piston-cylinder [2].

six methods, and second, to recommend a method for computing the effective area. In this note we do this and more: In Section 1.2 we give an introduction to the *virtual piston model* for piston-cylinder assemblies. This model itself is well known and well described in [1], on which most of the remaining sections are based. In Section 1.3 we show how, under certain simplifications, the model yields the various formulas used by the metrological institutes. In Section 1.4 we give a mathematically rigorous treatment of the Navier-Stokes equations for incompressible Newtonian fluids, which also lead to the same formula. Of course, our model itself still depends on certain simplifications, and in Section 1.5 we argue, at least for two of these simplifications, that there is no point in relaxing them, since that would only have higher order effects on the results. In Section 1.6 we comment on the computation of uncertainty limits; in Section 1.7 it is described how the formulas for the effective area should be evaluated in a numerical sound way and finally, in Section 1.8 we present the desired recommendations to NMi.

We start with a simplified description of the piston-cylinder unit used for the pressure measurement. Figure 1.1 shows the basic geometry of this device. It consists of a vessel containing a viscous fluid (air or oil) with a (nearly) cylindrical opening in which a piston can move up and down.

Inside the vessel, the fluid is under pressure $p_1 = p_2 + \Delta p$ where p_2 denotes the ambient pressure outside the device. A pressure measurement is done by the weight of the piston so that an equilibrium is reached between this weight and the forces exerted by the fluid on the piston. The largest part of this force results from the pressure acting from below onto the piston.

Between the piston and the surrounding cylinder, however, there is a narrow interstice in which a small amount of fluid is pressed upward. This leads to a frictional force exerted by the fluid to the flanks of the piston, and this force contributes to counterbalancing the weight of the piston.

The so-called *effective area* A_{eff} of the device is the area which would be needed in an idealized situation to counterbalance the weight W of the piston just from the pure pressure force:

$$A_{\text{eff}} := \frac{W}{\Delta p}.$$

Let l denote the length of the piston. We assume that both the piston P and the surrounding cylinder C are perfectly round, i.e., they are given by

$$\begin{aligned} P &:= \{(x, y, z) \,|\, z \in (0, L),\ x^2 + y^2 < r(z)\}, \\ C &:= \{(x, y, z) \,|\, z \in (0, L),\ x^2 + y^2 < R(z)\}, \end{aligned}$$

respectively.

Our crucial assumption here is that both R and r have small variations and that their difference $h := R - r$ is small compared to the radii:

$$\varepsilon := \frac{h}{r} \ll 1.$$

In practice, ε is of the order 10^{-4} to 10^{-5}. Hence, in the situation we are interested in, terms which are of order ε^2 (or higher) can safely be neglected.

Note that in Section 1.2 and in [1], a slightly differing approach to the concept of effective area is taken: The concept of a so-called *virtual piston* is introduced, consisting of the actual piston together with an annular column of liquid between the actual piston and the neutral surface between cylinder and piston at which no shear forces act inside the liquid. For this virtual piston, the friction force between the piston and the liquid is an internal force, and there is no need to calculate it explicitly. Now in [1] the effective area is defined as

$$S := \frac{W + w}{\Delta p},$$

where w is the weight of the annular liquid column. In our situation, however, including the gravitation force term in the lubrication equations (see Section 1.4) shows that $w/\Delta p$ is of order ε^2, therefore no difference to order ε exists between A_{eff} and S. Due to this fact, the different approach taken by SMU in their calculation of the effective area does not lead to essentially different results.

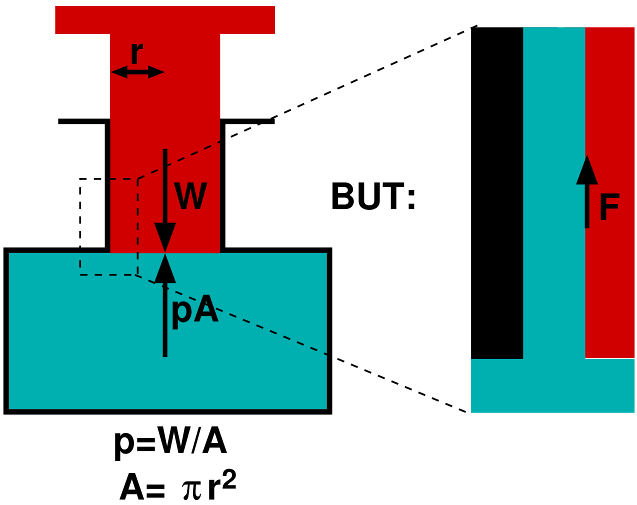

Figure 1.2: A piston-cylinder assembly.

1.2 The virtual piston model

First of all we give a gentle introduction to the *virtual piston model*, using the concept of a *virtual cylinder*. We are given a cylinder and a cylindrical piston of radius r moving in it. The piston has a certain weight W (which includes the so-called applied weights on top of the piston). An ambient buoyancy correction has to be done because of the ambient buoyancy effect on the submerged part of the floating component. This W depends, of course, on the gravity g, but we assume that it can be measured or computed very accurately. In the naive model, depicted on the right in Figure 1.2, the piston and the cylinder are perfect (vertical) cylinders with a perfect fit. In this case, when one knows the area A of the piston, the pressure p can be calculated from the force equilibrium

$$pA = W.$$

Hence it suffices to know, in addition to W, the *nominal area* $A = \pi r^2$ to calculate the pressure p.

However, as suggested on the left in Figure 1.2, there is a small gap between the piston and the cylinder, through which the medium moves upward, exerting an upward frictional force on the piston. Let R be the radius of the cylinder, and set $h := R - r$ and $\epsilon := h/r$. The parameter ϵ will always be assumed small, and in fact our formulas will be exact up to terms of order ϵ^2. To get rid of this frictional force, one defines the *neutral surface* between the cylinder and piston to be the surface where the velocity of the medium is maximal, and one replaces the piston by the *virtual piston*, which is the actual piston enlarged with the annular column of the medium bounded on the one side by the piston

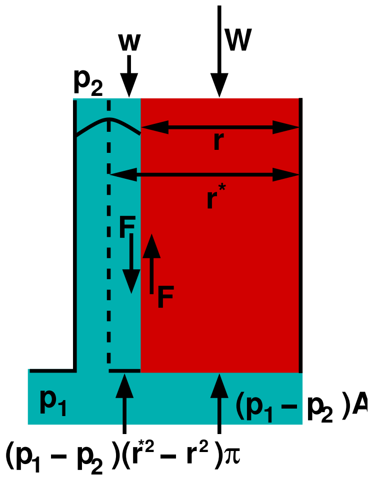

Figure 1.3: The neutral surface and the virtual piston.

and on the other side by the neutral surface—see Figure 1.3. The reason for working with this virtual piston is that no friction is exerted on it anymore: there is no friction among the layers of medium at the neutral surface. Let w be the weight of the annular column of medium between the piston and the neutral surface; again, we assume that w can be measured or calculated very accurately. Now the *effective area* A_{eff} of the piston-cylinder assembly is defined as *the area that would explain why the virtual piston of weight* $W + w$ *is in equilibrium with the pressure from below.* In a formula, we must have

$$A_{\text{eff}}(p_1 - p_2) = W + w,$$

where p_1 is the pressure below the piston and p_2 is the ambient pressure. Hence, to compute the pressure p_1 it suffices to know W, w, p_2 and A_{eff}.

If the piston and cylinder are still assumed perfect cylinders as in Figure 1.3, then the neutral surface is also a cylinder, whose radius we denote by r^*. It follows from the classical theory of viscous flow between cylindrical surfaces [4] that

$$(r^*)^2 = \frac{R^2 - r^2}{2log(\frac{R}{r})}. \tag{1.1}$$

Writing $R = r(1 + \epsilon)$, we get the following expansion for $(r^*)^2$.

$$(r^*)^2 = r^2(1 + \epsilon + \frac{\epsilon^2}{6} + O(\epsilon^3)).$$

In fact, r^* is equal to the arithmetic mean $(R+r)/2$ plus terms of order $O(\epsilon^2)$ due to the roundness of cylinder and piston. Other expressions that agree with $(R+r)/2$ up to terms of order ϵ^2 are the geometric mean $\sqrt{Rr}$ or $\sqrt{(R^2 + r^2)/2}$.

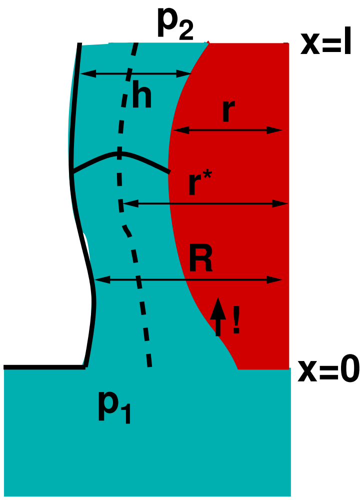

Figure 1.4: A non-perfect assembly.

All these expressions are used in the literature. The next combination of rR and $(R+r)/2$ gives a second order approximation for $\lambda = \frac{1}{6}$.

$$(r^*)^2 = 4\lambda(\frac{r+R}{2})^2 + (1-4\lambda)rR \doteq r^2(1+\epsilon+\lambda\epsilon^2). \tag{1.2}$$

In this perfect-cylinder case the formula for A_{eff} is easy:

$$A_{\text{eff}} = \pi(r^*)^2 = \pi((R+r)/2)^2 + O(\epsilon^2).$$

From the six European metrological institutes only NMi uses this formula. However, the piston-cylinder assemblies under consideration are not perfect. We do assume that they have perfect rotational symmetry around a vertical axis (see Model B in Section 1.5 for a discussion of this assumption). Then the piston and the cylinder are described by their radii r and R as a function of the vertical coordinate x; see Figure 1.4. The neutral surface will also have rotational symmetry, hence be given by its radius r^* as a function of $x \in [0, l]$. Furthermore, the pressure p is a function of x, as well, and so is h. Following [1] we sometimes write r_0, R_0, r_0^*, h_0 for the values of r, R, r^*, h at 0, and p_1, p_2 for $p(0), p(l)$.

Now the virtual piston has weight $W + w$, and this is in equilibrium with the following forces exerted on it:

1. A force equal to $\pi r^*(0)^2 p_1 - \pi r^*(l)^2 p_2$ due to the pressure working on both ends of the virtual piston, and

2. a force equal to $\int_0^l p(\xi) \frac{d\pi(r^*)^2}{dx}(\xi) d\xi$ due to the vertical component of the fluid pressure acting on the inclined flanks of the virtual piston.

Equilibrating these with $W + w$ and partial integration yields

$$\begin{aligned} W + w &= \pi r^*(0)^2 p_1 - \pi r^*(l)^2 p_2 + \int_0^l p(\xi)\frac{d\pi(r^*)^2}{dx}(\xi)d\xi \\ &= \pi r^*(0)^2 p_1 - \pi r^*(l)^2 p_2 + \left[p(\xi)\pi r^*(\xi)^2\right]_0^l - \int_0^l \pi r^*(\xi)^2 \frac{dp}{dx}(\xi)d\xi \\ &= -\int_0^l \pi r^*(\xi)^2 \frac{dp}{dx}(\xi)d\xi \end{aligned} \tag{1.3}$$

This formula has a nice intuitive interpretation: the infinitesimal pressure difference $-\frac{dp}{dx}$ at height ξ pushes upward against the circular horizontal cut at height ξ of the virtual piston; and all these forces together are in equilibrium with $W + w$.

Dividing by $p_1 - p_2$, we find that

$$A_{\text{eff}} = -(p_1 - p_2)^{-1} \int_0^l \pi r^*(\xi)^2 \frac{dp}{dx}(\xi)d\xi. \tag{1.4}$$

Now we will often use the geometric mean $\sqrt{Rr}$ as an approximation for r^*. Moreover, we introduce the two new variables $u := r - r_0$ and $U := R - R_0$, which are also assumed to be $O(\epsilon)$. Then $(r^*)^2 = rR + O(\epsilon^2) = (r-u)(R+u) + O(u) + O(\epsilon^2) = r_0(r_0 + h_0 + U + u) + O(u) + O(\epsilon^2)$. Substituting this approximation, we find that the effective area is approximately

$$A_{\text{eff}} \simeq \pi r_0^2 \left\{ 1 + \frac{h_0}{r_0} - \frac{1}{r_0(p_1 - p_2)} \int_0^l (u(\xi) + U(\xi))\frac{dp}{d\xi}d\xi \right\}. \tag{1.5}$$

Most European metrological institutes use equivalent or simplified versions of this formula. The goal is now, given r and R as functions of x (or rather, lists of their values measured at finitely many levels in $[0, l]$), and assuming a suitable model for the pressure p, to compute the effective area A_{eff} using the formula above.

1.3 Simplifications under further assumptions

Having determined the formula (1.4), it is still not possible to calculate the effective area of the piston: the formula contains the unknown pressure p_1 (which is to be determined!) and, even worse, the derivative $p'(\xi)$ of the pressure in the thin layer between the piston and the cylinder. In this section we show that from formula (1.5) one can, under additional assumptions, derive various other formulas in which all variables are known.

Since the annulus between the cylinder and the piston is very small ($h/r = \epsilon \ll 1$) the fluid motion in this gap is at zeroth order well described by the so-called thin film or lubrication approximation of the Navier-Stokes equation. For the derivation we use (again) the rotationally symmetric nature of the problem and the fact that the ratio h/r is small. These two features allow us to consider the problem as a 2D one and then apply the rotational symmetry to obtain a full 3D picture. The fluid in a vertical 2D slice has velocity $\mathbf{v} = (v_1, v_2)$, where $v_1(x, y)$ is the velocity component in the vertical x-direction and $v_2(x, y)$ the component in the horizontal y-direction. The equations are then

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 v_1}{\partial y^2}, \quad \frac{\partial p}{\partial y} = \mu \frac{\partial^2 v_2}{\partial y^2}, \quad \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0,$$

where μ is the viscosity, that is assumed independent of the pressure p. For a viscous fluid the natural boundary conditions are $v_1 = v_2 = 0$ on the walls, so on $y = 0 + u(x)$ and $y = h + U(x)$. In first approximation this yields the solution $v_2 \equiv 0$, $p = p(x)$, $v_1 = \frac{1}{2\mu}\frac{dp}{dx}(y - u)(y - h - U) \approx \frac{1}{2\mu}\frac{dp}{dx} y(y - h)$ if u and U are much smaller than h.

The fluid velocity flux Q through through a *horizontal* slice of the annulus is the fluid velocity integrated over this area. The rotational symmetry and small fraction h/r yield that this flux is at leading order

$$Q = 2\pi r \int_0^h v_2(y)dy = 2\pi r \frac{1}{2\mu}\frac{dp}{dx}\left[\frac{1}{3}y^3 - \frac{1}{2}hy^2\right]_{y=0}^{h},$$

which yields the formula

$$\frac{Q}{\pi r} = -\frac{1}{6\mu}\frac{dp}{dx}h^3. \tag{1.6}$$

Since the fluid in the annulus is a thin film between two metal side walls, the temperature of the fluid can be assumed constant, so that isothermic laws apply.

Assemblies operating with incompressible fluids

For incompressible fluids, the flux Q is constant. Since r is constant at leading order, this implies that the right-hand side of (1.6) is constant at leading order, so that $\frac{d}{dx}[-\frac{dp}{dx}h^3] = 0$. Integration leads to

$$p(x) = p_1 - (p_1 - p_2)\frac{\int_0^x \frac{1}{h(\xi)^3}d\xi}{\int_0^l \frac{1}{h(\xi)^3}d\xi} \tag{1.7}$$

and

$$\frac{dp}{dx} = -(p_1 - p_2)\frac{\frac{1}{h(x)^3}}{\int_0^l \frac{1}{h(\xi)^3}d\xi}. \tag{1.8}$$

Substitution into (1.4) gives the formula

$$A_{\text{eff}} = \frac{\int_0^l \pi r^*(\xi)^2 \frac{1}{h(\xi)^3}d\xi}{\int_0^l \frac{1}{h(\xi)^3}d\xi}. \tag{1.9}$$

This formula only contains variables that are known by measurements and interpolation between the measured data. It is, under the assumption of pressure-independent viscosity, valid for all values p_1, p_2. In other words, for incompressible fluids the resulting effective area is pressure-independent, which, of course, is what makes the effective area a useful characteristic of piston-cylinder assemblies! This formula and variations on it are used by IMGC, LNE, PTB, and UME. This seems reasonable for liquid-operated assemblies under not too high pressure, as under low pressure liquids in general behave as incompressible fluids. The formulas below for gas-operated assemblies look similar to (1.9), but are slightly more complicated. In particular, the constant

$$C := \int_0^l \frac{1}{h(\xi)} d\xi \tag{1.10}$$

will appear over and over again, and we will abbreviate it to C.

Gas-operated assemblies

For gas-operated assemblies, and also for liquid-operated assemblies under very high pressure, the assumption of incompressibility is no longer realistic. For such fluids it is no longer the flux Q, but the value $Q\rho$ that is constant, where ρ is the density. From (1.6) we then derive that

$$\frac{Q\rho}{\pi r} = -\frac{\rho}{6\mu}\frac{dp}{dx}h^3$$

is constant at leading order. According to the gas law $pV = mRT$ the quotient p/ρ is constant under isothermic conditions, so that $-\frac{p}{6\mu}\frac{dp}{dx}h^3$ is constant and has zero derivate as well. For pressure-independent viscosity integration now leads to

$$p(x) = \left[p_1^2 - \frac{p_1^2 - p_2^2}{C}\int_0^x \frac{1}{h(\xi)^3} d\xi\right]^{1/2}, \tag{1.11}$$

where C is the constant defined in (1.10); hence

$$\frac{dp}{dx}(x) = -\frac{p_1^2 - p_2^2}{2C}\frac{1}{h(x)^3}\left(p_1^2 - \frac{p_1^2 - p_2^2}{C}\int_0^x \frac{1}{h(\xi)^3} d\xi\right)^{-1/2}. \tag{1.12}$$

This formula can again be substituted in (1.4). The resulting effective area A_{eff} is no longer independent of p_1 and p_2 and can in theory not be determined as long as the pressure p_1 is unknown. However, A_{eff} is in fact just a function of the ratio p_1/p_2. Under the assumption that $\lim_{x\to\infty} A_{\text{eff}}(x)$ exists, this means in particular that $\lim_{p_2\to 0} A_{\text{eff}}(\frac{p_1}{p_2})$ is independent of the value p_1: if the assembly is immersed in vacuum, so with $p_2 \to 0$, the effective area is independent of p_1.

The institute PTB used the resulting expression for A_{eff} (their formula (3) plugged into their (2)), and then extrapolated for $p_1 - p_2 \to 0$.

Small applied pressure

The expression for A_{eff} for a compressible fluid has two limits, in which the formula becomes more attractive. If we assume that $p_1 \gg (p_1 - p_2)$, we consider the situation in which the pressure difference is small compared to the pressure p_1 (or p_2). Equivalently, one can consider the limit $p_2 \to p_1$. If, after substitution of (1.12) into (1.4), this limit is taken, then the result is precisely equation (1.9), the formula that gives the effective area in case of an incompressible fluid.

Thus one can conclude that under small pressure differences *any* fluid, compressible or incompressible, leads to the same effective area. This makes it even more attractive to use this formula and validates the choice of IMGC, LNE, PTB, and UME in a sense.

Large applied pressure

The other limit we take is the limit for large applied pressure, so $p_1 \gg p_2$. Since for compressible fluids A_{eff} is a function of p_1/p_2, the limit $p_2 \to 0$ describes this situation. In this limit (1.12) reduces to

$$\frac{dp}{dx}(x) = -\frac{p_1}{2C}\frac{1}{h(x)^3}\left(1 - \frac{\int_0^x \frac{1}{h(\xi)^3}d\xi}{C}\right)^{-1/2},$$

which in turn leads to an effective area

$$A_{\text{eff}} = \int_0^l \frac{\pi r^*(x)^2}{2Ch(x)^3}\left(1 - \frac{\int_0^x \frac{1}{h(\xi)^3}d\xi}{C}\right)^{-1/2} dx. \tag{1.13}$$

In deriving these formulas we implicitly assumed that the linear (isothermic) gas law is still valid for these high pressure conditions. The limit (1.13) thus obtained is useful for gas-operated assemblies with high $p_1 - p_2$. Note that the effective area is (again) independent of the values p_1 and p_2, but differs from the effective area for the low applied pressure or incompressible case. Note also that it involves computing a double integral, where the bound x of the inner integral is the variable of the outer integral; this makes numerical evaluation of the expression above rather awkward.

1.4 The Navier-Stokes equations for incompressible fluids

After the rather informal approach using the virtual piston model, we will now derive formula (1.9) more rigorously, making precise what simplifications of reality underly the model.

The motion of air of oil between the inner r and outer radius R can be described by the Navier-Stokes equations for incompressible Newtonian fluids, given by [5, 3]

$$\rho\Big(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\Big) = -\nabla p + \mu\nabla^2\boldsymbol{v} - \rho g\boldsymbol{e}_x, \qquad \nabla\cdot\boldsymbol{v} = 0, \tag{1.14}$$

where ρ, $\boldsymbol{v}$, p and g denote density, velocity, pressure and the gravitational acceleration. This expression is valid for a uniformly constant viscosity μ. This appears to be a reasonable assumption, as the large heat capacity of the metal cylinder is probably able to absorb any generated heat and to keep the temperature, and thus the viscosity, of the fluid constant.

In view of the geometry of piston and cylinder, we choose cylindrical coordinates (r, ϕ, x), while v, w, u will denote the r, ϕ, x component of the velocity $\boldsymbol{v}$. Note the difference between r and r: the latter is, as always, the radius of the piston as a function of x, while the former is the radial coordinate! The stationary problem becomes in axial, radial and circumferential components

$$\rho\left(v\frac{\partial u}{\partial \mathrm{r}} + \frac{w}{\mathrm{r}}\frac{\partial u}{\partial \phi} + u\frac{\partial u}{\partial x}\right) = \mu\left(\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}\frac{\partial u}{\partial \mathrm{r}}\right) + \frac{1}{\mathrm{r}^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial x^2}\right) - \frac{\partial p}{\partial x} - \rho g, \tag{1.15a}$$

$$\rho\left(v\frac{\partial v}{\partial \mathrm{r}} + \frac{w}{\mathrm{r}}\frac{\partial v}{\partial \phi} - \frac{w^2}{\mathrm{r}} + u\frac{\partial v}{\partial x}\right) = \mu\left(\frac{\partial}{\partial \mathrm{r}}\left(\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}(\mathrm{r}v)\right) + \frac{1}{\mathrm{r}^2}\frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial x^2} - \frac{2}{\mathrm{r}^2}\frac{\partial w}{\partial \phi}\right) - \frac{\partial p}{\partial \mathrm{r}}, \tag{1.15b}$$

$$\rho\left(v\frac{\partial w}{\partial \mathrm{r}} + \frac{w}{\mathrm{r}}\frac{\partial w}{\partial \phi} + \frac{vw}{\mathrm{r}} + u\frac{\partial w}{\partial x}\right) = \mu\left(\frac{\partial}{\partial \mathrm{r}}\left(\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}(\mathrm{r}w)\right) + \frac{1}{\mathrm{r}^2}\frac{\partial^2 w}{\partial \phi^2} + \frac{\partial^2 w}{\partial x^2} + \frac{2}{\mathrm{r}^2}\frac{\partial v}{\partial \phi}\right) - \frac{1}{\mathrm{r}}\frac{\partial p}{\partial \phi}, \tag{1.15c}$$

$$\frac{\partial u}{\partial x} + \frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}(\mathrm{r}v) + \frac{1}{\mathrm{r}}\frac{\partial w}{\partial \phi} = 0. \tag{1.15d}$$

Both the slowly sinking piston and the rotation can be completely modeled by the boundary conditions! It is convenient to combine p and ρg into the reduced pressure

$$\bar{p} = p + \rho g x. \tag{1.16}$$

When we scale the axial velocity on a typical (as yet unknown) velocity U, the radial velocity on hU/l, the circumferential velocity on the given rotational velocity, say U/δ (where δ is small), radial derivatives on the typical width $h = R - r$, radial distance r and axial derivatives on the slit length l, the circumferential derivatives on a small paramete γ, the (reduced) pressure on $\mu U l/h^2$, while we call the small parameter $\varepsilon = h/l$ and the Reynolds number in axial direction $Re = \rho U h/\mu$. Notice that $\varepsilon \neq \epsilon = \frac{h}{r}$ but has the same order of magnitude. Then we get in dimensionless form

$$Re\varepsilon\left(v\frac{\partial u}{\partial \mathrm{r}}+\frac{\gamma}{\delta}\frac{w}{\mathrm{r}}\frac{\partial u}{\partial \phi}+u\frac{\partial u}{\partial x}\right)=$$
$$\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}\frac{\partial u}{\partial \mathrm{r}}\right)+\gamma^2\varepsilon^2\frac{1}{\mathrm{r}^2}\frac{\partial^2 u}{\partial \phi^2}+\varepsilon^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial \bar{p}}{\partial x}, \qquad (1.17a)$$

$$Re\varepsilon^2\left(\varepsilon v\frac{\partial v}{\partial \mathrm{r}}+\frac{\varepsilon\gamma}{\delta}\frac{w}{\mathrm{r}}\frac{\partial v}{\partial \phi}-\frac{1}{\delta^2}\frac{w^2}{\mathrm{r}}+\varepsilon u\frac{\partial v}{\partial x}\right)=$$
$$\varepsilon^2\left(\frac{\partial}{\partial \mathrm{r}}\left(\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}(\mathrm{r}v)\right)+\gamma^2\varepsilon^2\frac{1}{\mathrm{r}^2}\frac{\partial^2 v}{\partial \phi^2}+\varepsilon^2\frac{\partial^2 v}{\partial x^2}-\frac{\varepsilon\gamma}{\delta}\frac{2}{\mathrm{r}^2}\frac{\partial w}{\partial \phi}\right)-\frac{\partial \bar{p}}{\partial \mathrm{r}}, \qquad (1.17b)$$

$$Re\varepsilon\left(v\frac{\partial w}{\partial \mathrm{r}}+\frac{\gamma}{\delta}\frac{w}{\mathrm{r}}\frac{\partial w}{\partial \phi}+\varepsilon\frac{vw}{\mathrm{r}}+u\frac{\partial w}{\partial x}\right)=$$
$$\frac{\partial}{\partial \mathrm{r}}\left(\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}(\mathrm{r}w)\right)+\gamma^2\varepsilon^2\frac{1}{\mathrm{r}^2}\frac{\partial^2 w}{\partial \phi^2}+\varepsilon^2\frac{\partial^2 w}{\partial x^2}+\gamma\delta\varepsilon^3\frac{2}{\mathrm{r}^2}\frac{\partial v}{\partial \phi}-\frac{\gamma\delta}{\mathrm{r}}\frac{\partial \bar{p}}{\partial \phi}, \qquad (1.17c)$$

$$\frac{\partial u}{\partial x}+\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}(\mathrm{r}v)+\frac{\gamma}{\delta}\frac{1}{\mathrm{r}}\frac{\partial w}{\partial \phi}=0. \qquad (1.17d)$$

Thus the order of magnitude estimates of the both sides of the equations equal

$$Re\varepsilon, Re\varepsilon\gamma/\delta, Re\varepsilon = 1, \gamma^2\varepsilon^2, \varepsilon^2, 1, \qquad (1.18a)$$
$$Re\varepsilon^3, Re\varepsilon^3\gamma/\delta, Re\varepsilon^2/\delta^2, Re\varepsilon^3 = \varepsilon^2, \gamma^2\varepsilon^4, \varepsilon^4, \varepsilon^3\gamma/\delta, 1, \qquad (1.18b)$$
$$Re\varepsilon, Re\varepsilon\gamma/\delta, Re\varepsilon^2, Re\varepsilon = 1, \gamma^2\varepsilon^2, \varepsilon^2, \gamma\delta\varepsilon^3, \gamma\delta, \qquad (1.18c)$$
$$1, 1, \gamma/\delta = 0. \qquad (1.18d)$$

So if ε is small, with $Re \preceq \mathcal{O}(\varepsilon)$, $\gamma^2 \preceq \mathcal{O}(\frac{1}{Re}\varepsilon^4)$, $\delta^2 \succeq \mathcal{O}(Re)$ and $\gamma\delta \preceq \mathcal{O}(\varepsilon^2)$ then all small terms are equal to or smaller than $\mathcal{O}(\varepsilon^2)$, and we are left with

$$\frac{\mu}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}\frac{\partial u}{\partial \mathrm{r}}\right)-\frac{\partial \bar{p}}{\partial x}=0, \qquad (1.19a)$$
$$\frac{\partial \bar{p}}{\partial \mathrm{r}}=0, \qquad (1.19b)$$
$$\frac{\partial}{\partial \mathrm{r}}\left(\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}\frac{\partial w}{\partial \mathrm{r}}\right)\right)=0, \qquad (1.19c)$$
$$\frac{\partial u}{\partial x}+\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}(\mathrm{r}v)+\frac{\gamma}{\delta}\frac{1}{\mathrm{r}}\frac{\partial w}{\partial \phi}=0. \qquad (1.19d)$$

All this is to be verified a posteriori, because the order of magnitude of U is unknown yet. Equation (1.19a) is the most important equation here, and known as Reynold's lubrication equation. Equation (1.19b) says that the pressure only depends on x. Equation (1.19c) says that circumferential velocity component w is decoupled from the rest of the problem, so it can be ignored as it doesn't contribute to the pressure difference between top and bottom. Equation (1.19d) relates v and w to u, but can also be ignored for the present problem.

The inner cylinder is slowly but steadily moving down by its own weight W, and we assume at time t the bottom to be at height x_p with (constant) velocity u_p, given by

$$x = x_p(t), \qquad u_p = \frac{\mathrm{d}x_p}{\mathrm{d}t}. \tag{1.20}$$

The position of the inner cylinder is conveniently described by

$$\mathrm{r} = r(x - x_p). \tag{1.21}$$

The boundary conditions along the cylindrical surfaces $\mathrm{r} = r$ and $\mathrm{r} = R$, taking into account the same approximation as before by ignoring all $\mathcal{O}(\varepsilon^2)$-terms, become [7]

$$u = u_p \quad \text{at} \quad \mathrm{r} = r(x - x_p), \tag{1.22a}$$

$$u = 0 \quad \text{at} \quad \mathrm{r} = R(x). \tag{1.22b}$$

Conservation of mass requires that as much mass is squeezed out of the cavity as corresponds to the incoming volume of the inner cylinder [7]:

$$2\pi \int_r^R u(\mathrm{r}, x)\mathrm{r}\,\mathrm{dr} = -\pi u_p r^2. \tag{1.23}$$

Note that the above expression is the volume flux at height x. This is not the same for every x, because the slit width h may vary with x.

The total force on the inner cylinder [7] is now given by the pressure difference between top and bottom (multiplied by the respective areas) plus the shear and normal stresses of the flow in the slit. Re-expressed in terms of $\bar{p}$ this is given by

$$\begin{aligned} F &= 2\pi \int_{x_p}^{x_p+l} \left[\bar{p}r'r + \mu\frac{\partial u}{\partial \mathrm{r}}r\right]_{\mathrm{r}=r} \mathrm{d}x + \pi\rho g \int_{x_p}^{x_p+l} r^2(x - x_p)\,\mathrm{d}x \\ &\qquad + \pi\left[r^2(0)\bar{p}(x_p) - r^2(l)\bar{p}(x_p + l)\right] \\ &= \pi \int_{x_p}^{x_p+l} \left[-\frac{\mathrm{d}\bar{p}}{\mathrm{d}x}r^2 + 2\mu\frac{\partial u}{\partial \mathrm{r}}r\right]_{\mathrm{r}=r} \mathrm{d}x + \pi\rho g \int_0^l r^2(s)\,\mathrm{d}s. \end{aligned} \tag{1.24a}$$

From equations (1.19a, 1.22a, 1.22b) and (1.23) we have

$$2\mathrm{r}\mu\frac{\partial u}{\partial \mathrm{r}} = \mathrm{r}^2\frac{\mathrm{d}\bar{p}}{\mathrm{d}x} - \frac{\frac{1}{2}\frac{\mathrm{d}\bar{p}}{\mathrm{d}x}(R^2 - r^2) + 2\mu u_p}{\log(R/r)} \tag{1.25a}$$

$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}x} = \frac{4\mu u_p}{(R^2 + r^2)\log(R/r) - (R^2 - r^2)} \tag{1.25b}$$

(Note that velocity u_p is as yet unknown.) This leads to the total force on the inner cylinder to be given by

$$F = -2\pi\mu u_p \int_0^l \frac{R^2 + r^2}{(R^2 + r^2)\log(R/r) - (R^2 - r^2)}\,\mathrm{d}s + \pi\rho g \int_0^l r^2\,\mathrm{d}s. \tag{1.26}$$

The unknown velocity u_p is obtained from the condition that for a steady situation the force F should be equal to the weight of the cylinder W. So we have

$$u_p = -\frac{W - \pi\rho g \displaystyle\int_0^l r^2 \,\mathrm{d}s}{2\pi\mu \displaystyle\int_0^l \frac{R^2 + r^2}{(R^2 + r^2)\log(R/r) - (R^2 - r^2)} \,\mathrm{d}s}. \tag{1.27}$$

This yields all the information necessary to determine the pressure difference between top and an bottom. If u_p is known it is also possible to estimate the value of U. From (1.23) it follows that $\pi(R^2 - r^2)U \approx \pi|u_p|r^2$, so

$$U = \mathcal{O}(\frac{|u_p|}{2\varepsilon}).$$

If $|u_p| \preceq \mathcal{O}(\varepsilon^2)$ the previous assumption that $Re \preceq \mathcal{O}(\varepsilon)$ is correct. Because of (1.31a) and (1.31b) we can estimate that

$$u_p \simeq \frac{W}{6\pi\mu l}\varepsilon^3.$$

If we use the following estimates (for air):

$$\begin{aligned}
r = l &= 6 \text{ cm},\\
W &= 5000 \text{ g},\\
\mu &= 1.78 \cdot 10^{-4} \text{ g/cm s},\\
\rho &= 1.2 \cdot 10^{-3} \text{ g/cm}^3,\\
\varepsilon &= 5 \cdot 10^{-5}
\end{aligned}$$

we obtain

$$\begin{aligned}
u_p &= 3 \cdot 10^{-8} \text{ cm/s},\\
U &= 3 \cdot 10^{-4} \text{ cm/s},\\
Re &= 6 \cdot 10^{-7},
\end{aligned}$$

so indeed $Re \ll \varepsilon$. The order condition $\gamma^2 \preceq \mathcal{O}(\frac{1}{Re}\varepsilon^4)$ is fulfilled if $\gamma \leq 3 \cdot 10^{-6}$, a very small number. For $\delta \sim 8 \cdot 10^{-4}$ the side-effects can be neglected because then $\delta^2 \succeq \mathcal{O}(Re)$ and $\gamma\delta \preceq \mathcal{O}(\varepsilon^2)$. Because $\delta = U/2\pi r f \approx 7.96 \cdot 10^{-6}/f$, it follows that the rotational frequency $f \sim 10^{-2}$ rev/s $= 0.6$ rev/min. This result is different from the results of Michels [1], who found much higher critical speeds lying generally within the range 28 to 32 rev/min. This diference could be explained by the fact that we used different parameter values. However, it is also mentioned in [1] that there is evidence that considerably lower speeds are quite practical with well-made piston-cylinder assemblies.

We have

$$\bar{p}(x_p) - \bar{p}(x_p + l) = -\int_{x_p}^{x_p+l} \frac{\mathrm{d}\bar{p}}{\mathrm{d}x}\,\mathrm{d}x \tag{1.28}$$

Thus we get

$$p(x_p) - p(x_p + l) = \rho g l + \frac{2}{\pi}\left(W - \pi\rho g \int_0^l r^2 \mathrm{d}s\right) \frac{\displaystyle\int_0^l \frac{1}{(R^2+r^2)\log(R/r) - (R^2 - r^2)}\mathrm{d}s}{\displaystyle\int_0^l \frac{R^2+r^2}{(R^2+r^2)\log(R/r) - (R^2 - r^2)}\mathrm{d}s}. \tag{1.29}$$

This is a complete and, within the theory of lubrication flow with slowly varying walls [5] and moderate Reynolds number, exact result. We can make considerable progress, however, by using the fact that the slit is not only slowly varying but also very close to, and very thin compared to, a typical cylinder radius. We choose a fixed radius R_{eff}, which will be chosen in a convenient way and which will correspond to the effective area, and introduce

$$r(s) = R_{\text{eff}} - h_1(s), \tag{1.30a}$$
$$R(s) = R_{\text{eff}} + h_2(s), \tag{1.30b}$$
$$h(s) = h_1(s) + h_2(s), \tag{1.30c}$$
$$R(s) = r(s) + h(s), \tag{1.30d}$$

where h_1 and h_2 are both of the same order of magnitude as h. Then we can approximate for small h

$$(R^2+r^2)\log(R/r) - (R^2 - r^2) = \frac{2h^3}{3R_{\text{eff}}} + (h_1 - h_2)\frac{h^3}{3R_{\text{eff}}^2} + \mathcal{O}(h^5/R_{\text{eff}}^3), \tag{1.31a}$$
$$R^2 + r^2 = 2R_{\text{eff}}^2 - 2R_{\text{eff}}(h_1 - h_2) + \mathcal{O}(h^2). \tag{1.31b}$$

This yields the rather unwieldy expression

$$p(x_p) - p(x_p + l) \simeq \rho g l + \left(\frac{W}{\pi R_{\text{eff}}^2} - \rho g l + \frac{2\rho g}{R_{\text{eff}}}\int_0^l h_1 \mathrm{d}s\right) \frac{\displaystyle\int_0^l \frac{1}{h^3} - \frac{1}{2}\frac{h_1 - h_2}{R_{\text{eff}} h^3}\mathrm{d}s}{\displaystyle\int_0^l \frac{1}{h^3} - \frac{3}{2}\frac{h_1 - h_2}{R_{\text{eff}} h^3}\mathrm{d}s}. \tag{1.32}$$

A clever choice of R_{eff}, however, is the one which makes

$$\int_0^l \frac{h_1 - h_2}{R_{\text{eff}} h^3}\mathrm{d}s = 0. \tag{1.33}$$

This is achieved by

$$R_{\text{eff}} = \frac{\displaystyle\int_0^l \frac{R+r}{h^3}\mathrm{d}s}{\displaystyle 2\int_0^l \frac{1}{h^3}\mathrm{d}s} \tag{1.34}$$

Notice that R_{eff} can be viewed as the radius of a generalized *neutral surface.* In this case our expression greatly simplifies to

$$p(x_p) - p(x_p + l) \simeq \frac{W}{\pi R_{\text{eff}}^2} + \frac{2\rho g}{R_{\text{eff}}} \int_0^l h_1 \mathrm{d}s \tag{1.35}$$

This can be interpreted as the well-known effective area, see [1] and Section 1.1. If we define

$$A_{\text{eff}} = \pi R_{\text{eff}}^2 \tag{1.36}$$

and note that

$$w = 2\pi R_{\text{eff}} \rho g \int_0^l h_1 \mathrm{d}s \tag{1.37}$$

is (to the order of approximation) equal to the weight of the cylinder of fluid between R_{eff} and r, then

$$A_{\text{eff}}(p(x_p) - p(x_p + l)) \simeq W + w \tag{1.38}$$

In conclusion: the systematic and most general definition of effective area, for piston-cylinder assemblies operating with incompressible fluids, is given by equations (1.34) with (1.36). Up to order ϵ^2, this approach leads to the same expression as formula (1.9) in Section 1.3.

1.5 Further side effects

The model (1.4) which has been derived in section 1.2, is based on a lot of assumptions.

- The piston and cylinder are axisymmetric.
- The vertical velocity of the piston is zero in the stationary case.
- The system converges sufficiently fast to the stationary state.
- There is no rotation because the stationary case is stable.
- The piston and cylinder have the same axis.
- The elastic properties of the material of the piston and cylinder are not important.
- The temperature variations because of the friction can be neglected.

In practice these assumptions are not fulfilled, as we now explain. We will shortly describe the physical aspects of the piston-cylinder unit and enumerate the side-effects which are not modelled by Dadson's theory, which is described in Section 1.2.

There is a fluid/gas below the piston and also between the walls of the piston and cylinder. In what follows, we will concentrate on the case of incompressible fluids. We start from an initial state, for which there is no fluid between the walls. Because of the gravity force the piston will sink rather fast. Fluid will flow between the moving walls, which implies an upward force as reaction on the gravity force. This upward force will grow when the piston sinks until both forces are equal (stationary case). Because this equilibrium should be unstable, the piston is rotating with fixed angular frequency around its fixed axis. Because of the viscosity the cylinder will also rotate. It follows that the walls of the piston and cylinder do not touch each other.

The first side-effect is the fact that the radii of the piston and cylinder depend on z and ϕ. Second, the piston falls with a constant speed in the stationary case. In Dadson's theory it was assumed that this speed is zero but this is not always true. We are interested in the stationary case for which the piston falls with this constant speed and is still rotating. A third side-effect is that the piston is rotating in order to get rid of the instability.

In section 1.4 a general formula for the effective area is given. It is directly derived from the Navier-Stokes equations for incompressible Newtonian fluids. The model includes the fact that the piston is slowly sinking. Furthermore, conditions are given, such that the model can be assumed to be axisymmetiric.

In [2, 8] one considers finite element models which also include the elasticity of the material of the piston and the cylinder. If we take care with moving axes we should also consider the dry friction forces if the cylinder and piston touch each other. In [1] it is shown how to deal with the rotation and the moving axes. It has been shown that the resultant of the viscous forces is zero by symmetry.

We will consider the following two extended models.

A This model assumes that the piston and cylinder are perfect cylinders around the same axis. There is no rotation. We only consider the effect that the piston slowly sinks in the stationary case.

B This model assumes that there is no rotation and it is assumed that the piston does not sink in the stationary case. We only assume that the radius of the piston and cylinder depends on z and ϕ.

Model A: sinking of piston

Consider the piston and cylinder of constant radius r and R. We are interested in the stationary case where the upward wet friction force is equal to the downward gravity force. Note that the force have to be corrected because of the buoyancy force on the piston below the fluid level. For moving axes it has been proved in [1] that the exact value of r^* satisfies

$$(r^*)^2 = r^2(1 + \epsilon + \frac{7}{12}\epsilon^2 + O(\epsilon^3)). \qquad (1.39)$$

Thus the influence of the sinking piston is $O(\epsilon^2)$.

Model B: variable radius for piston and cylinder

Assume that the radii depend on z and ϕ. Then the neutral surface will also depend on z and ϕ! It is very hard, to compute $r^*(z,\phi)$ in an analytical way. It is defined as the radius of the virtual cylinder between the piston and cylinder for which the force between adjacent layers of fluid will be zero. This means that there the tangential component of the force is zero! Note that formula (1.4) can be written as

$$A_{\text{eff}}(p_2 - p_1) = \int_0^l \pi (r^*)^2 \frac{dp}{dx} dx.$$

Because now r^* and $\frac{dp}{dx}$ also depend on ϕ we get

$$A_{\text{eff}}(p_2 - p_1) = \int_0^l \int_0^{2\pi} \pi (r^*)^2 \frac{dp}{dx} d\phi dx. \tag{1.40}$$

In [1] it is stated that p satisfies the two-dimensional continuity equation:

$$\frac{\partial}{\partial z}\left\{h^3 \frac{\rho}{\mu}\frac{\partial p}{\partial z}\right\} + \frac{\partial}{\partial \phi}\left\{h^3 \frac{\rho}{\mu}\frac{\partial p}{\partial \phi}\right\} = 6\left\{U\frac{\partial}{\partial z}(\rho h) + V\frac{\partial}{\partial \phi}(\rho h)\right\}, \tag{1.41}$$

where z and ϕ are the axial and circumferential coordinates and U and V are the relative velocities of the two surfaces in the axial and circumferential directions respectively.

From practice it follows that the non-roundness is of the same order as the measurement errors. This implies that it indeed can be assumed that the piston and cylinder are axisymmetric.

1.6 Uncertainty limits

Standard uncertainty of measurements

In all measurements, we have to deal with measurement errors. These are usually modeled as normally distributed uncertainties $\Delta(x_i)$, that are superimposed to the 'real' values of each measurement x_i. The standard uncertainty of measurement x_i is then defined as the standard deviation of $\Delta(x_i)$, which we denote by $\sigma(x_i)$.

For every measuring instrument, some standard uncertainty of measurement is specified, which can be used to calculate the overall uncertainty of some physical entity A, that is derived from the measurements x_i.

If all measurement errors are independent from each other, we can use the following first order approximation to calculate the overall uncertainty in A :

$$\sigma(A)^2 = \sum_{i=1}^{n}\left(\sigma(x_i)\frac{\partial A}{\partial x_i}\right)^2$$

The piston measurement uncertainties

The NMi has provided us with piston and cylinder measurement data and their standard uncertainties. The piston diameter was measured at 13 different heights (ξ-coordinates), with a standard uncertainty of 50 nm, which is determined by the standard uncertainty of the measuring equipment. However, the sample standard deviation of these 13 measurements is only 14 nm. Even if there are only small fluctuations in the 'real' diameter of the piston, we would expect a sample standard deviation of at least 50 nm. The fact that we find a so much smaller sample standard deviation can not be attributed to chance. No matter what statistical test we apply, we allways find p-values smaller than 10^{-10}. The same phenomenon appears in the piston measurements that were performed at all other metrology institutes.

We conclude that the standard uncertainties of both piston and cylinder measurements must have a systematic and a random component. Both components are unknown, but the systematic component is always the same for all measurements, whereas the random components are independent from one another. This kind of situation can occur for instance in mass measurements, where the unknown mass is compared to a standard mass. This standard mass has some unknown deviation from the exact value. All measurements performed with the same standard mass will therefore have a systematic error equal to the deviation of the standard mass.

When confronted with systematic uncertainties, we have to adapt the formula for the first order approximation of the overall uncertainty of some physical entity A:

$$\sigma(A)^2 = \sum_{i=1}^{n} \left(\sigma(x_i)\frac{\partial A}{\partial x_i}\right)^2 + \left[\sum_{i=1}^{n} \left(\widetilde{\sigma}(x_i)\frac{\partial A}{\partial x_i}\right)\right]^2, \tag{1.42}$$

where $\sigma(x_i)$ is the random (uncorrelated) component and $\widetilde{\sigma}(x_i)$ is the systematic component of the uncertainty in the measurement of x_i.

It is also possible to model the uncertainties in a more general way, by introducing certain correlations between every pair of seperate measurements. This will however lead to quite complicated mathematical models. Another major drawback of this approach is in the fact that it is very hard (if not impossible) to make good estimates for these correlations.

Numerical calculation of the propagation of measurement errors

In this section we give an example of how the propagation formula (1.42) can be handled in a comprehensive numerical way.

We make two assumptions in the computation of uncertainties. First, we assume to deal with the case of an incompressible fluid. Second, we suppose that the approximation $\tilde{S}$ for the effective area A_{eff}, is obtained by replacing the integrals by Riemann sums (i.e. the integrand is approximated by a staircase function).

These choices are not very restrictive. Indeed, from the analytic formulas, it is derived in section 1.3 that the formula for incompressible as well as the one for compressible fluids coincide in the limit of small applied pressure (i.e. in case $p_1 - p_2 \ll p_1$, that we are dealing with). Hence the restriction to the case of an incompressible fluid does not imply loss of generality. Concrete, we assume that the effective area is expressed by formula (1.43):

$$A_{\mathrm{eff}} = \pi r_0^2 + \pi r_0 h_0 + \pi r_0 \frac{\int_0^l \left(u\left(x\right) + U\left(x\right)\right) h\left(x\right)^{-3} \mathrm{d}x}{\int_0^l h\left(x\right)^{-3} \mathrm{d}x} \tag{1.43}$$

Furthermore, we calculate the uncertainty of measurement when the integrals are approximated by Riemann sums. Although it is a primitive approximation technique, it is the root of most other techniques when approximating integrals. As a consequence, the formula (1.44) derived below, can serve as a first order approximation of the random component of uncertainty of the effective area in general.

The Riemann sum approximation is obtained by dividing the interval $[0, l]$ first in n subintervals of equal length, say $[x_i, x_{i+1}], 0 \le i \le n-1$ with $x_0 = 0$ and $x_{i+1} - x_i = l/n$, Then, the expression in the right-hand side of (1.43) can be approximated by

$$\begin{aligned}
\tilde{S} &= \pi r_0^2 + \pi r_0 h_0 + \pi r_0 \frac{\sum_{i=0}^n \left(u\left(x_i\right) + U\left(x_i\right)\right) h\left(x_i\right)^{-3} \frac{l}{n}}{\sum_{i=0}^n h\left(x_i\right)^{-3} \frac{l}{n}} \\
&= \pi r_0^2 + \pi r_0 h_0 + \pi r_0 \frac{\sum_{i=0}^n \left(u_i + U_i\right) h_i^{-3}}{\sum_{i=0}^n h_i^{-3}} \\
&= \tilde{S}\left(r_0, R_0, r_1, R_1, \ldots, r_n, R_n\right)
\end{aligned}$$

where $u_i = u\left(x_i\right), U_i = U\left(x_i\right), h_i = h\left(x_i\right), \forall 0 \le i \le n$; recall also that $h_i = R_i - r_i, \forall 0 \le i \le n$. Let us denote by $\sigma\left(y\right)$ the random component of uncertainty of y and by $\tilde{\sigma}\left(y\right)$ the systematic component of uncertainty of y. Then, the random component of uncertainty of the effective area A_{eff} is defined by

$$\begin{aligned}
\sigma\left(A_{\mathrm{eff}}\right)^2 = &\left(\sigma\left(r\right) \frac{\partial \tilde{S}}{\partial r_0}\right)^2 + \left(\sigma\left(R\right) \frac{\partial \tilde{S}}{\partial R_0}\right)^2 + \sum_{i=0}^n \left(\sigma\left(r\right)^2 + \sigma\left(R\right)^2\right) \left(\frac{\partial \tilde{S}}{\partial h_i}\right)^2 \\
&+ \left(\tilde{\sigma}\left(r\right) \sum_{i=0}^n \frac{\partial \tilde{S}}{\partial r_i}\right)^2 + \left(\tilde{\sigma}\left(R\right) \sum_{i=0}^n \frac{\partial \tilde{S}}{\partial R_i}\right)^2 .
\end{aligned} \tag{1.44}$$

The partial derivatives that are encountered in (1.44) can be computed as follows. Put $\forall 0 \le i \le n$

$$C_i = -3\pi r_0 \cdot \frac{\sum_{j=0}^n h_j^{-3} \left[\left(u_i + U_i\right) - \left(u_j + U_j\right)\right]}{h_i^4 \left(\sum_{j=0}^n h_j^{-3}\right)^2}.$$

Then,

$$\frac{\partial \tilde{S}}{\partial r_0} = \pi R_0 + \pi \frac{\sum_{i=0}^{n} (u_i + U_i)\, h_i^{-3}}{\sum_{i=0}^{n} h_i^{-3}} - C_0;$$

$$\frac{\partial \tilde{S}}{\partial R_0} = \pi r_0 + C_0$$

$$\frac{\partial \tilde{S}}{\partial h_0} = \pi r_0 + C_0 \text{ and } \frac{\partial \tilde{S}}{\partial h_i} = C_i, \forall 1 \le i \le n;$$

$$\frac{\partial \tilde{S}}{\partial r_i} = -\frac{\partial \tilde{S}}{\partial h_i} \text{ and } \frac{\partial \tilde{S}}{\partial R_i} = \frac{\partial \tilde{S}}{\partial h_i}, \forall 1 \le i \le n.$$

If higher order integration methods are used (like Simpson's rule), every term in the Riemann sum will receive its own coefficient and nothing else will change. Therefore, the same approach can still be applied to the calculation of the partial derivatives.

1.7 Numerical implementation of Dadson's formula

For incompressible fluids the formula (1.9) has been shown in section 1.3 to be a proper approximation of the effective area. The following equivalent formulas are used by four institutes [6]. They are all equivalent and can be derived from (1.9).

$$A_{\text{eff}} = \pi r_0^2 \left\{ 1 + \frac{1}{r_0} \frac{\int_0^l \frac{1}{h^2} dx}{\int_0^l \frac{1}{h^3} dx} + \frac{2}{r_0} \frac{\int_0^l \frac{u}{h^3} dx}{\int_0^l \frac{1}{h^3} dx} \right\} \tag{1.45}$$

$$A_{\text{eff}} = \pi r_0^2 \left\{ 1 + \frac{h_0}{r_0} + \frac{1}{r_0} \frac{\int_0^l \frac{u+U}{h^3} dx}{\int_0^l \frac{1}{h^3} dx} \right\} \tag{1.46}$$

$$A_{\text{eff}} = \pi r_0 \left\{ -r_0 + \frac{\int_0^l \frac{r+R}{h^3} dx}{\int_0^l \frac{1}{h^3} dx} \right\}. \tag{1.47}$$

The integrands of the integrals are continuous functions which have to be approximated by use of the measurements. It is possible to create the integrand-functions themselves directly by interpolation or to create the functions $r, R : [0, l] \to \mathbb{R}^+$ first. Therefore the cylinder and piston radii are measured for $z = z_i$, resulting in the set $\{(R_i, r_i), i = 1, \ldots, N\}$. The grid $\{z_i, i = 1, \ldots, N\}$ of the length axis of the piston-cylinder unit can be used to control the accuracy of the resulting continuous functions in an adaptive way. If r, R behave very smoothly it is more efficient to use higher order interpolation, while low order interpolation is better for less smooth surfaces. Furthermore linear interpolation could be used in order to conserve the monotonicity.

Each integral can be evaluated with a numerical integration technique, like Newton-Cotes or Gaussian quadrature formulas. The most straightforward numerical integration technique uses the Newton-Cotes formulas (also called quadrature formulas), which approximate a function tabulated at a sequence of regularly spaced intervals by various degree polynomials. Common Newton-Cotes formulas include the Trapezoidal Rule (Linear), Simpson's Rule (Parabolic) and Simpson's 3/8 Rule (Cubic) . If the functions are known analytically instead of being tabulated at equally spaced intervals, the best numerical method of integration is called Gaussian quadrature, which uses non-uniformly spaced grid points. Common Gaussian quadratures include the Gauss-Legendre Formula and the Gauss-Chebyshev Formula. It could be more efficient to use an adaptive grid, which is more dense where $h(x) \approx 0$. Also it can be synchronized with the grid $\{z_i, i = 1, \ldots, N\}$ of the measurements in order to minimize the interpolation errors. The Newton-Cotes formulas are less accurate but significantly less complicated to implement.

If we compare the three unscaled formulas we see that no cancellation errors occur. In all cases the denominator can become very small if the clearance h tends to zero. Therefore we have to scale the variables in order to avoid serious trouble because of roundoff errors. Write $h = \epsilon_h \bar{h}, u = \epsilon_u \bar{u}$ and $U = \epsilon_U \bar{U}$, where $\bar{h}, \bar{u}, \bar{U}$ are $O(1)$. Then the formulas (1.45),(1.46),(1.47) can be written as

$$A_{\text{eff}} = \pi r_0^2 \left\{ 1 + \frac{\epsilon_h}{r_0} \frac{\int_0^l \frac{1}{\bar{h}^2} dx}{\int_0^l \frac{1}{\bar{h}^3} dx} + \frac{2\epsilon_u}{r_0} \frac{\int_0^l \frac{\bar{u}}{\bar{h}^3} dx}{\int_0^l \frac{1}{\bar{h}^3} dx} \right\}, \tag{1.48}$$

$$A_{\text{eff}} = \pi r_0^2 \left\{ 1 + \epsilon_h \frac{\bar{h}_0}{r_0} + \frac{1}{r_0} \frac{\int_0^l \epsilon_u \frac{\bar{u}}{\bar{h}^3} + \epsilon_U \frac{\bar{U}}{\bar{h}^3} dx}{\int_0^l \frac{1}{\bar{h}^3} dx} \right\}, \tag{1.49}$$

$$A_{\text{eff}} = \pi r_0 \left\{ -r_0 + \frac{\int_0^l \frac{r+R}{\bar{h}^3} dx}{\int_0^l \frac{1}{\bar{h}^3} dx} \right\}. \tag{1.50}$$

Formulas (1.48) and (1.49) have the advantage that the effective area is expressed in the zeroth order term πr_0^2 and two first order corrections. From literature [9] it appears that the piston shape deviations are much smaller than the cylinder shape deviations, which implies that $\epsilon_u \ll \epsilon_U$. Therefore it is recommended to use the scaled formula (1.48).

1.8 Recommendations

In this document we have shown several models for the piston-cylinder unit. First, there is the *virtual piston model* which has a very useful form, which is given in (1.4). The models in [1] and [6] are specific cases of this model. Second, a formula for the effective area has been derived directly from the Navier-Stokes equations for incompressible fluids. Then it is even possible to get exact results. However, we also proved that the formula (1.9) is sufficiently accurate for incompressible fluids, because the error is of order ϵ^2, where ϵ is a small number.

Clearly one should always make use of the fact that one can *measure* the dimensions of the piston and cylinder with much higher accuracy than can be achieved in the *production* process. Therefore we recommend NMi to use a more advanced model for the effective area, like the first order approximation (1.9). It is also used by four other European institutes. A sound numerical integration method should be used like described in Section 1.7. Finally we advise to make a distinction between systematic errors and random errors, which makes it possible to get much sharper uncertainty bounds.

1.9 Bibliography

[1] R.S. Dadson, S.L. Lewis, and G.N. Peggs. *The Pressure Balance: Theory and Practice*. HMSO, London, UK, 1982.

[2] T.J. Esward, R.C. Preston, and P.N. Gélat. Piston-cylinder pressure balance design with a negligible distortion coefficient. *Meas.Sci.Technol.*, 14:796–806, 2003.

[3] A.C. Fowler. *Mathematical models in the applied sciences*. Cambridge University Press, 1997.

[4] H. Lamb. *Hydrodynamics*. Cambridge, University Press, UK, 1932.

[5] R.M.M. Mattheij, S.W. Rienstra, and J.H.M ten Thije Boonkkamp. *Partial Differential Equations: Modeling, Analysis, Computation*. SIAM, Philadelphia, 2005.

[6] G. Molinar, M. Bergoglio, W. Sabuga, P. Otal, G. Ayyildiz, J. Verbeek, and P. Farar. Calculation of effective area A_0 for six piston-cylinder assemblies of pressure balances. Results of the EUROMET Project 740. *Metrologia*, 42:197–201, 2005.

[7] S.W. Rienstra and T.D. Chandra. Analytical approximations to the viscous glass flow problem in the mould-plunger pressing process, including an investigation of boundary conditions. *Journal of Engineering Mathematics*, 39:241–259, 2001.

[8] N.D. Samaan. *Mathematical modelling of instruments for pressure metrology*. PhD thesis, City University, London, 1990.

[9] J. Verbeek. NMi-DH350_rad, 2004.