Will the ringing of the Bourdon bell damage the Old Church Delft?

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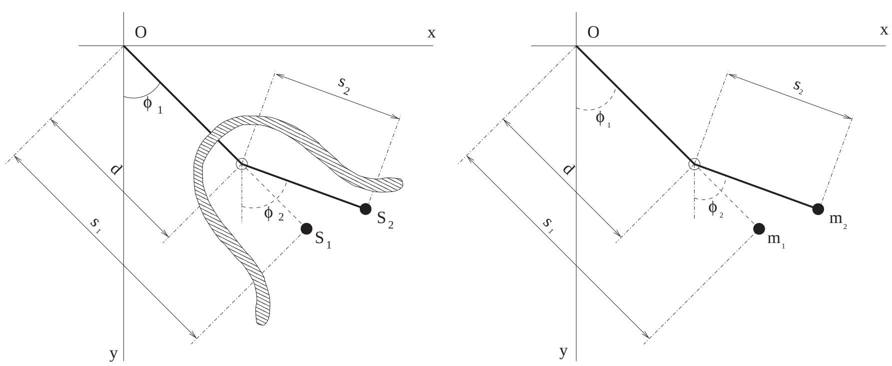
1. Introduction

The first parish church of Delft, the old church, was built around 1200. In front of the church a 75 meters high tower, with brickwork spire and four turrets, was built in 1350. Even during its construction, the tower was plagued by subsidence. This could be because the water in the Oude Delft had to be redirected to make way for the existing church. The tower therefore was probably built on a filled-in canal. Throughout the ages, the leaning tower has been the cause of considerable alarm to many an inhabitant. The tower leans 1.20 meters to the west and 1 meter to the north.
Two unique bells hang from a heavy oak bell cage in the fourth loft in the tower of the Oude Kerk (Old Church). These are the Trinitas bell dating from 1570 and the Laudate bell dating from 1719. The Trinitas bell, or Bourdon bell, is the most exceptional of the two, weighing almost nine tonnes . The Bourdon is only rung on very special occasions such as, for example, the funeral of a member of the Dutch royal family. The powerful chime of the Bourdon causes such heavy vibrations that regular use could damage the monument.
In this paper we tried to model the effect of ringing the extremely heavy bells inside the leaning tower with mathematical methods. We modelled the leaning Old Chuch by a skew vertical Euler-Bernoulli beam and we modelled the swinging of the Bourdon bell as a non-linear singular pendulum. First in section 2 we will consider the bell model and calculate the force acting on the tower due to the swinging bell. In section 3 we will consider the skew beam model and calculate the maximum displacement in both leaning directions due to the swinging bell.

2. Mathematical model for the bell

2.1. Introduction. This section describes a mathematical model for the swing of the Bourdon bell in the tower of the Old Church in Delft. Although a bell with clapper (or bob) is formally a coupled system of 2 pendulums, we will use a single physical pendulum approximation for the computation of the pendulum period because of the relatively small mass of the clapper compared with the bell itself.
Nevertheless, as the amplitude is relatively high (about ± 70 degrees) we do not use a linear approximation such as the harmonic oscillator but use elliptic integrals to describe this pendulum in a more accurate way.
Figure (1) shows the double pendulum model and notations while figure (2) shows the single pendulum approximation.



(a) Bourdon bell with bob (b) Generalized double pendulum

FIGURE 1. Bell and bob parameters

2.2. Analysis for a non-linear singular pendulum. Using Newton's second law for rotational motion on this pendulum yields as usual for the balance of moments:

$$I_0\ddot{\varphi} = -m\ g\ s\sin(\varphi) \tag{1}$$

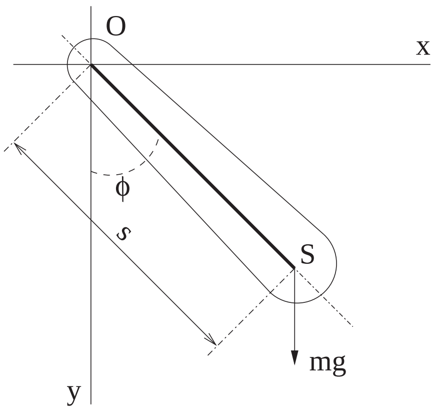
Here I_0 denotes the moment of inertia, ϕ the amplitude, m the mass, s the distance to the centre of gravity and g the gravity constant (see also fig. 2).

Substituting the commonly used "radius of gyration" i:

$$I_0 = m\ i^2 \tag{2}$$

yields:

$$\ddot{\varphi} + \frac{g\ s}{i^2}\sin(\varphi) = 0 \tag{3}$$



I_0 = moment of inertia around O

m = mass of pendulum

S = center of gravity

FIGURE 2. Single pendulum approximation (schematic)

With the angular speed ω defined by:

$$\omega = \frac{\sqrt{g\ s}}{i} \tag{4}$$

we obtain the standard form for the non-linear pendulum:

$$\ddot{\varphi} + \omega^2 \sin(\varphi) = 0 \tag{5}$$

We take as initial conditions (at $t = 0$):

$$\varphi(0) = \varphi_{max} = \alpha \text{ and } \dot{\varphi}(0) = 0$$

If we multiply (5) with $\dot{\varphi}$ and integrate we find:

$$\frac{1}{2}\int \frac{d}{dt}(\dot{\varphi})^2 = \omega^2 \int \frac{d}{dt}(\cos(\varphi + C))$$

From the initial conditions we can replace the value for C:

$$\dot{\varphi}^2 = 2\omega^2\{\cos(\varphi) - \cos(\alpha)\} \tag{6}$$

Using $1-\cos(\varphi) = 2\sin^2(\frac{\varphi}{2})$ and $1-\cos(\alpha) = 2\sin^2(\frac{\alpha}{2})$ and introducing the parameter k according to:

$$k = \sin(\frac{\alpha}{2})\ ,\ 0 \leq k \leq 1 \tag{7}$$

we can rewrite (6) into:

$$\dot{\varphi}^2 = 4k^2\omega^2(1 - \frac{\sin^2(\varphi/2)}{k^2}) \tag{8}$$

Introduction of a new variable $y = \frac{\sin(\varphi/2)}{k}$ transforms the problem into equation (10) with the help of:

$$\dot{\varphi}^2 = \frac{4k^2(\dot{y})^2}{1 - k^2y^2} \tag{9}$$

$$\dot{y} = \omega\sqrt{(1-y^2)(1-k^2y^2)} \tag{10}$$

Finally this yields as solution (11):

$$\omega\, t + C = \int_0^y \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}} \tag{11}$$

2.3. Elliptic integrals. The integral in the righthand side of (11) is a so-called elliptic integral of the first kind in the Legendre normal form and it's solution is available in a tabulated form ([**2**]).
It has an inverse $sn(x)$ on $0 \leq y \leq 1$, which is an elliptic function of Jacobi and where $sn(x)$ is a periodic function with a $4K$ period.
$K(k)$ is defined as follows:

$$K(k) = \int_0^1 \frac{d\eta}{\sqrt{(1-\eta^2)(1-k^2\eta^2)}} \tag{12}$$

So, on the interval $0 \leq y \leq 1$ $sn(K(k)) = k$ and $sn(K) = 1$. From (11) it follows that the solution can be written as $y = sn(\omega t + C)$. Also $sn(0) = 0$ and with $k \to 0$ we obtain $sn(x) \to \sin(x)$.
Now putting all pieces together we find:

$$y = \frac{\sin(\phi/2)}{k} = sn(\omega t + C) \tag{13}$$

With the initial conditions from above at $t = 0$ ($\phi(0) = \alpha$), we obtain

$$\sin\frac{\alpha}{2} = k\ sn(C), \text{ or: } sn(C) = 1, \text{ and } C = K$$

Finally, we find the solution

$$\begin{aligned} \sin\frac{\phi}{2} &= ksn(\omega\ t + K) \\ \text{and: } \phi(t) &= 2\arcsin\{sn(\omega\ t + K)\} \end{aligned} \tag{14}$$

We need the relation between $K(k)$ and k on the interval $< 0, 1 >$ before we are able to see how frequency changes with the bell amplitude. We can search a few points and present the relation between k and $K(k)$ roughly in a figure:
From (12) we find for $k = 0$:

$$K(0) = \int_0^1 \frac{d\eta}{\sqrt{1-\eta^2}} = \arcsin 1 = \pi/2$$

Also, because the integral diverges for $k \uparrow 1$ we have $\lim_{k\to 1}\ K(k) = \infty$.
For $\alpha = 90^o$ or $\pi/2$ (remember α denotes ϕ_{max}) $k = \sin\alpha/2 = 0.707$.
We can find $K(0.707)$ numerically or by looking it up in a table such as in [**2**]. Either way we obtain $K(0.707) \approx 1.854$.
These values were used to sketch fig. (3).

For a more accurate approximation we have to do some extra work:

From (14) it follows, since the period of sn is $4K$, that the period of the pendulum equals:

$$T = \frac{4K}{\omega}$$

The influence of the nonlinearity can be estimated from:

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

obtained by substituting $\eta = \sin\theta$ in the integral from (12).

An expansion of $(1 - k^2 \sin^2 \theta)^{-1/2}$ in the binomial series gives:

$$(1 - k^2 \sin^2 \theta)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-k^2)^n \sin^{2n} \theta$$

Integrating term by term and using:

$$\int_0^{\pi/2} \sin^{2n}(\theta) d\theta = 2^{-2n} \binom{2n}{n} \frac{\pi}{2}$$

it follows that:

(15)

$$K(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \binom{-1/2}{n} \binom{2n}{n} (-k/2)^{2n} = \frac{\pi}{2}\{1 + \frac{k^2}{4} + \frac{9k^4}{64} + \dots\}$$

So for the period T we find:

(16) $$T = \frac{4K}{\omega} = \frac{2\pi}{\omega}\{1 + \frac{k^2}{4} + \frac{9k^4}{64} + \dots\}$$

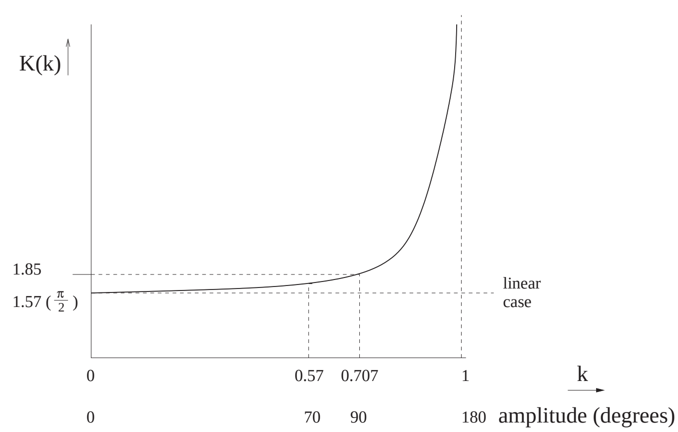


FIGURE 3. Relation between k and $K(k)$

Bell:		
Height	1820 mm	Royal Eijsbouts
Mass (without crown)	7700 kg	Royal Eijsbouts
Centre of Gravity (from bottom bell)	728 mm	Royal Eijsbouts
Moment of inertia in CG (S)	5110 $kg.m^2$	Royal Eijsbouts
Rotation axis (from bottom bell)	1460 mm	TNO Delft
Crown:		
Mass	550 kg	Royal Eijsbouts
Distance from rotation axis	300 mm	estimated
Counterweight:		
Mass	1000 kg	estimated
Distance from rotation axis	1100 mm	TNO Delft / estimated

TABLE 5. Some properties of the Bourdon bell

where the first term $2\pi/\omega$ represents the period of the linearized system (the harmonic oscillator!).

For $k = 0.707$ (or the amplitude $\alpha = \frac{\pi}{2}$), it follows by using the first 2 terms:

$$k^2/4 + 9k^4/64 \approx 0.16$$

So for an amplitude of $\frac{\pi}{2}$ the nonlinear terms raise T with 16% as compared to the linearized case.

$K(0.707) \approx \frac{\pi}{2}\,(1+0.16) \approx 1.83$ whereas 1.854 is the value from ([**2**] we already used above.

For k close to 1 (amplitude $\alpha = \pi$) the series converges only slowly and there we need more and more terms to find a reasonable accurate approximation, but for the Bourdon bell this is not necessary.

2.4. Computing the bell period. To be able to use the formulas above we need detailed values for many properties of the Bourdon bell. Unfortunately some of the values below are at this moment only rough estimations.
The Royal Eijsbouts Company and TNO Delft (both in the Netherlands) provided us with some numbers. Other were estimated by ourselves using photographs. The most important ones are listed in table (5).

If we assume that the maximum amplitude of the Bourdon bell $\alpha = 70^o(= 7/18\pi)$ we have $k = \sin(\frac{\alpha}{2}) = 0.57$. Using the formula from (15)

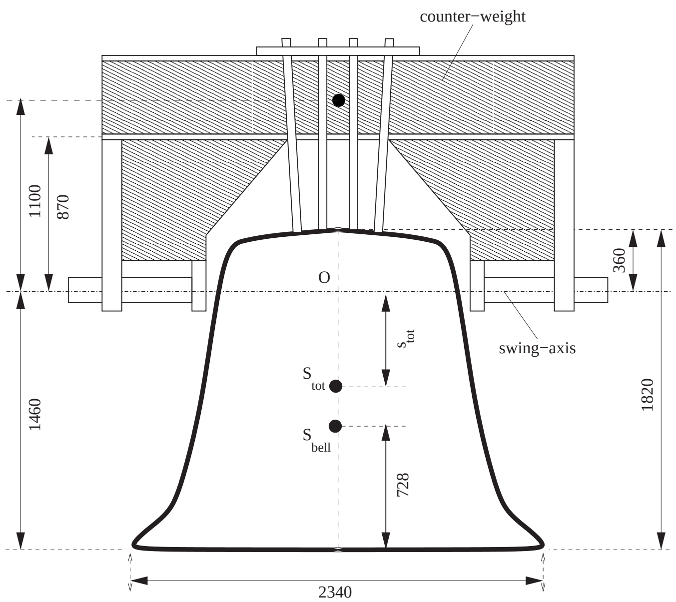


FIGURE 4. Bourdon bell dimensions

or the graph from figure (3) we find:

$$K(0.57) \approx \frac{\pi}{2}\{1 + \frac{0.57^2}{2} + 9\frac{0.57^4}{64}\} = 1.73$$

The Bourdon bell has a counterweight on the opposite side of the rotation axis O (see also figure (4)). This implies that the new center of gravity S_{tot} for the total combined system must be shifted towards O. This new center can be easily computed:

$$\begin{aligned} M_{counter}.(S_{counter} + s_{tot}) &= M_{bell}.(S_{bell} - s_{tot}) \\ 1000.(1100 + s_{tot}) &= 7700.(732 - s_{tot}) \\ s_{tot} &= 4536400/8700 = 521\ mm\ (0.52\ m) \end{aligned}$$

M_{tot} bell + crown + counterbalance: 9250 kg
I_{bell} around O (using Steiners rule): $5110 + 7700.(732)^2 = 9235\ kg.m^2$

Total moment of inertia:

$$\begin{aligned} I_0 &= I_{bell} + I_{counter} + I_{crown} \\ &= 9235 + 1000.(1.1)^2 + 550.(0.03)^2 \\ &\approx 10500\ kg.m^2 \end{aligned}$$

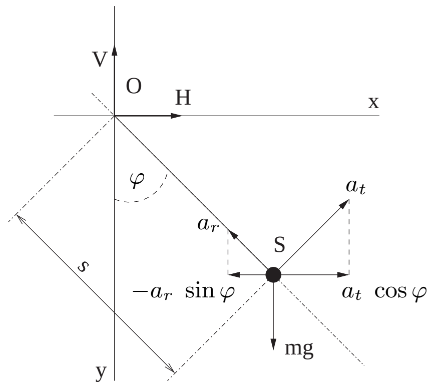


FIGURE 5. Contribution of accelerations to force (here only shown for H)

Now we can gather all other numbers and compute the period:

$$
\begin{aligned}
i &= \sqrt{\frac{I_0}{M}} = \sqrt{\frac{10500}{9250}} \approx 1.07\ m \\
\omega &= \frac{\sqrt{g\ s_{tot}}}{i} = \frac{\sqrt{9.81.0.52}}{1.07} \approx 2.11\ rad/sec. \\
T &= \frac{4\ K}{\omega} = \frac{4.1.73}{2.11} \approx \mathbf{3.28\ sec}
\end{aligned}
$$

As there are about 2 chimes per period (the clapper hits the bell twice per period) this implies $n = \frac{2}{T}.60 = \frac{2\ .\ 60}{3.28} \approx 37$ chimes/min.

Linear approximation: If we would have used the linear approach (simple harmonic oscillator) the result would be $T_{lin} = \frac{2\Pi}{\omega} = 2.98$ and 40 chimes/min.
So the difference is $\frac{K(0)}{K(\alpha)} = \frac{1.57}{1.73} = 0.91$ or about 9 %.

2.5. The forces on the axis of rotation. The acceleration of the centre of gravity S consists of 2 components (see also figure 5):

$$
\begin{aligned}
\text{in radial direction} &: \ a_r = s\dot{\varphi}^2 \\
\text{in tangential direction} &: \ a_t = s\ddot{\varphi}
\end{aligned}
$$

So using the contributions of both in the x-direction we find:

$$
\begin{aligned}
\ddot{x}_s &= -a_r \sin\varphi + a_t \cos\varphi \\
&= -s\ \dot{\varphi}^2 \sin\varphi + s\ddot{\varphi}\cos\varphi \\
&= -s\ \omega^2 \sin\varphi\{4(k^2 - \sin^2(\frac{\varphi}{2})) + \cos\varphi\}
\end{aligned}
$$

The last line was obtained by using equations (5) and (8)).

With $\sum \vec{F}_x = m\vec{\ddot{x}}_s$ the horizontal force H on the pendulum in the reference point 0 (to the rightside) is given by:

$$H = -m\ s\ \omega^2\{4(k^2 - \sin^2(\frac{\varphi}{2}))\sin\varphi + \sin\varphi cos\varphi\} \tag{17}$$

and with $\sum \vec{F_y} = m\vec{\ddot{y}_s}$ the vertical force V in the reference point 0 (in upwards direction) is given in a similar way by:

$$\ddot{y}_s = a_r\cos\varphi + a_t\sin\varphi = s\ \omega^2\{4(k^2 - \sin^2(\frac{\varphi}{2}))\cos\varphi - \sin^2\varphi\}$$

This yields by using the same substitutions as for the horizontal force:

$$V = m\ s\ \omega^2\{4(k^2 - \sin^2(\frac{\varphi}{2}))\cos\varphi - \sin^2\varphi\} + m\ g \tag{18}$$

Here of course φ is defined by equation (5).

By taking the derivative for H and V and see where these are equal to 0 we obtain the maximum amplitudes for H and V. Doing this in Maple yields:

$$\begin{aligned} V_{max} \text{ occurs for } \quad \phi = &\quad 0\ rad. \\ H_{max} \text{ occurs for } \quad \phi \approx &\quad 0.7\ rad. \end{aligned}$$

Using the numerical values we found in the previous section this yields for the forces:

$$\begin{aligned} H &\approx 23500\ N \\ V &\approx 115000\ N \text{ (or 30300 N without the } m\ g \text{ component)} \end{aligned}$$

Note that the signs of the forces are relative to the directions as shown in figure (5).

2.6. Conclusion. Given the fact that for several parameters we only have a very rough estimation (especially for the counterweight), the error caused by a linear approximation of the system is after all not such a big issue and may be even less than the error caused by incorrect parameters ...

The bell is only tolled at very special occasions of which the most important one is a funeral of a member of the Royal Dutch family, which is of course not a very common occasion.
However, just one day after the workshop ended the former Queen of the Netherlands Juliana died unexpectedly and so the bell was tolled at her funeral only a couple of days later.
We were able to obtain a short video recording from the swinging bell at the funeral, so we could measure the period straight from this video.

This turned out to be **3.2 sec** or about 37 chimes/min, so notwithstanding the possibly incorrect parameters the results from above resemble reality quite well!

3. The forces acting on the tower

The Old Church Tower consists of 4 parts with total length 75 meters. The bell of the tower is located at the height of 44 meters.The rest upper part of the tower has a much smaller weight than the previous parts. So we decide to take into account the influence of this part on the behavior of the tower construction as an additional weight which is uniformly distributed onto other parts. Different parts of the tower have different areas of cross sections and subsequently with different moments of inertia. The beam-like model can be applied for a first rough estimation of the displacement of the construction. In shipbuilding such model is often applied for the estimation of the hull behavior under the action of its weight and for vibration calculations. Because of the leaning the forces which are acting on the tower construction depend on the angle of leaning. The weight of the tower can be modeled as a load which is distributed according to the linear dependence on the length of the tower. The weight has two projections: one is along the tower axis and another perpendicular to the axis. In the process of ringing the dynamic force appears as a result of an action of a moving bell. This force also has two projections which change its values harmonically in time. So, finally the tower is modelled as a Euler-Bernoulli beam under the action of a linearly distributed weight along the length and a dynamic force caused by the bell.
For calculation of the eigen frequencies nevertheless another model of a beam should be introduced, because the ratio of the tower length to its width is 1:4.4 and the shear and longitudinal forces should be taken into account in computations of natural frequencies.
The boundary conditions for such a model unfortunately cannot be found exactly, because there is no data concerning the properties of the tower foundation and the soil properties. The boundary conditions which were considered are: one edge is fixed and another is free. For that type of boundary conditions and for fixed distribution of external forces, the displacements of the construction will be largest from all possible real boundary conditions. It is a special task to find out what boundary conditions should be taken into account for the real tower. In the next section, the governing equation for the tower structure will be described.

3.1. Beam equation. As a first approximation we model the skew tower by a skew Euler-Bernouilli beam:

$$(EI(x)u_{xx}(x,t))_{xx} + (T(x,t)u_x(x,t))_x + \rho A u_{tt}(x,t) = q(x,t)/L, \tag{19}$$

where $EI(x)$ is the bending stiffness, ρ the density, A the area of the beam, $T(x,t)$ the longitudinal compressible force due to the acceleration due to gravity(g) and the bell dynamic force and where $q(x,t)$ is the dynamic force. The longitudinal compression force is given by $W(x) = g_x[(1-x)u_x(x,t)]_x$ where if the tower has a leaning of α degrees, $g_x = g\cos(\alpha)$ and $g_y = g\sin(\alpha)$ and where $W(x)$ is the mass of the beam along its length and $u(x,t)$ the displacement of the beam along its length.
The governing system of equations has the following form:

$$(EI_1 u_{1_{xx}}(x,t))_{xx} + (T_1(x,t)u_{1_x}(x,t))_x + \rho A_1 u_{1_{tt}}(x,t) = q_1(x,t), \tag{20}$$
$$(EI_2 u_{2_{xx}}(x,t))_{xx} + (T_2(x,t)u_{2_x}(x,t))_x + \rho A_2 u_{2_{tt}}(x,t) = q_2(x,t), \tag{21}$$
$$(EI_3 u_{3_{xx}}(x,t))_{xx} + (T_3(x,t)u_{3_x}(x,t))_x + \rho A_3 u_{3_{tt}}(x,t) = q_3(x,t), \tag{22}$$
$$u_1(0,t) = u_{1_x}(0,t) = 0, \tag{23}$$
$$u_{3_{xx}}(L,t) = u_{3_{xxx}}(L,t) = 0. \tag{24}$$

with the following conditions at $x = L_1$ and $x = L_2$:

$$u_i(L_1,t) = u_j(L_i,t), \tag{25}$$
$$u_{i_x}(L_i,t) = u_{i+1_x}(L_i,t), \tag{26}$$
$$EI_i u_{i_{xx}}(L_i,t) = EI_{i+1}u_{i+1_{xx}}(L_i,t), \tag{27}$$
$$EI_i u_{i_{xxx}}(L_i,t) = EI_{i+1}u_{i+1_{xxx}}(L_i,t). \tag{28}$$

where $i = 1$ and $i = 2$ and where EI_i, A_i, T_i and $q_i(x,t)$ for $i = 1,2,3$ are the bending stiffness, the area the longitudinal compression force and the dynamic force of the three parts of the beam respectively.
Assume that $I(x)$ is a constant, then introduce the dimensionless variables $x = \frac{x}{L}$, $u(x,t) = \frac{u(x,t)}{L}$ and $t = t\sqrt{\frac{\rho A_2 L^4}{EI}}$, to put the equations (20)-(28) in the following non-dimensional form

$$u_{i_{xxxx}}(x,t) + \epsilon_i\left(\hat{W}(x)u_{i_x}(x,t)\right)_x + u_{i_{tt}}(x,t) = \alpha\epsilon_i\hat{W}(x,t), \tag{29}$$

where $i = 1,2,3$ and where $\epsilon_i = \frac{g_x L^3 \rho A_2}{EI_i}$, $\alpha = \frac{g_y}{g_x}$ and

$$\hat{W}(x) = \begin{matrix} \frac{3}{4} + \frac{A_1}{A_2}\left(\frac{1}{4} - x\right), 0 \le x \le \frac{1}{4}, \\ \frac{1}{4} + \left(\frac{3}{4} - x\right), \frac{1}{4} \le x \le \frac{3}{4}, \\ 1 - x, \frac{3}{4} \le x \le 1. \end{matrix} \tag{30}$$

3.2. The static problem. First we consider the static problem. The static equations describing the motion of the church are given by

$$u_{1_{xxxx}}(x) + \epsilon_1 \left(\hat{W}(x) u_{1_x}(x)\right)_x = \alpha \epsilon_1 \hat{W}(x), \tag{31}$$

$$u_{2_{xxxx}}(x) + \epsilon_2 \left(\hat{W}(x) u_{2_x}(x)\right)_x = \alpha \epsilon_2 \hat{W}(x), \tag{32}$$

$$u_{3_{xxxx}}(x) + \epsilon_3 \left(\hat{W}(x) u_{3_x}(x)\right)_x = \alpha \epsilon_3 \hat{W}(x), \tag{33}$$

$$u_1(0) = u_{1_x}(0) = u_{1_{xx}}(1) = u_{1_{xxx}}(1) = 0, \tag{34}$$

with the following conditions at $x = L_1 = \frac{1}{4}$ and $x = L_2 = \frac{3}{4}$

$$u_i(L_i, t) = u_{i+1}(L_i, t), \tag{35}$$

$$u_{i_x}(L_i, t) = u_{i+1_x}(L_i, t), \tag{36}$$

$$EI_i u_{i_{xx}}(L_i, t) = EI_{i+1} u_{i+1_{xx}}(L_i, t), \tag{37}$$

$$EI_i u_{i_{xxx}}(L_i, t) = EI_{i+1} u_{i+1_{xxx}}(L_i, t). \tag{38}$$

where $i = 1$ and $i = 2$.
If we have the values of ϵ_i and α we can determine the maximum static displacement of the beam. The tower is leaning in two directions (y and z). The first leaning is $2.5m$ and the second leaning is $0.5m$. So

$$\begin{aligned} \alpha_y &= \frac{g_x}{g_y} = \frac{2.5}{44} = 0.056, \\ \alpha_z &= \frac{g_x}{g_y} = \frac{0.5}{44} = 0.011. \end{aligned}$$

If the mass of the tower is $10^7 kg$ and the density is for every part is the same. The value for the product $\rho A_2 L$ for the tower is $0.94176 * 10^7 kg$. For the other properties we have

$$E = 2.5 * 10^9 Pa, \tag{39}$$

$$L = 44m, \tag{40}$$

$$g_x = 9.8 m/s^2, \tag{41}$$

$$I_1 = 1900 m^4, \tag{42}$$

$$I_2 = 1510 m^4, \tag{43}$$

$$I_3 = 1080 m^4. \tag{44}$$

Then we have $\epsilon_1 = 0.0376$, $\epsilon_2 = 0.0473$ and $\epsilon_3 = 0.0655$. The maximum displacements in the static case in y-direction and z-direction are given by

$$\begin{aligned} u_{max,static,y} &= 3.3mm, \\ u_{max,static,z} &= 0.67mm, \end{aligned}$$

respectively, where the subscript y, z denote the direction of the leaning. So the maximum displacement t of the tower is

$$u_{max,static} = \sqrt{u^2_{max,static,y} + u^2_{max,static,z}} = 3.4mm. \tag{45}$$

3.3. The dynamic problem. Now we consider the dynamic problemand so we also include the dynamic forces due to the swinging of the bell. We have the following partial differential equation for the ith part$(i = 1, 2, 3)$ of the beam

$$\begin{aligned} u_{i_xxxx}(x,t) + \epsilon_i \Big((\hat{W}(x) &+ B_1 \cos(\alpha)\cos(\omega t)) u_{i_x}(x,t) \Big)_x \\ &= \alpha \epsilon_i \hat{W}(x) + B_2 \sin(\alpha)\cos(\omega t) \end{aligned} \tag{46}$$

where $B_1 \cos(\alpha) = \frac{\hat{B}_1}{g\rho AL} = 0.0010$, $B_2 \sin(\alpha) = \frac{\hat{B}_2 L^2}{EI} = 5.1 * 10^{-5}$ and $\omega = \hat{\omega}\sqrt{\frac{\rho A_2 L^4}{EI}} = 0.50$. Here we used that $\hat{B}_1 = -1.1 * 10^5 N$, $\hat{B}_2 = -7.0 * 10^3 N$ and $\hat{\omega} = 2.18$.

Now we will consider the dynamic problem. For the dynamic problem we consider the three parts of the beam to have the same moment of inertia and area, the moment of inertia of the second part(i.e. $I = 1510m^4$). It is possible to use this very rough estimation. We suppose $u(x,0) = u_t(x,0) = 0$. So, we consider the following initial-boundary value problem describing the oscillations of the Old Churh due to the swiging of the bell:

$$\begin{aligned} u_{xxxx}(x,t) + \epsilon \left((\gamma + (1-x)) + \epsilon^2 b_1) u_x(x,t) \right)_x &+ u_{tt}(x,t) \\ &= \hat{\alpha}\epsilon^2(1-x) + \epsilon^4 b_2 \cos(\omega t), \end{aligned} \tag{47}$$

$$u_(0,t) = u_x(0,t) = u_{xx}(1,t) = 0, \tag{48}$$

$$\epsilon^2 \delta u_x(1,t) + \delta \epsilon u_{tt} - u_{xxx}(1,t) = 0, \tag{49}$$

$$u(x,0) = u_t(x,0) = 0. \tag{50}$$

The value for ϵ we considered in the previous section and is given by $\epsilon_2 = 0.047$. So, other parameters can be written as a product of ϵ. In this way the parameters b_1, b_2 and $\hat{\alpha}$ can be calculated and are given by $\delta = 1.6$, $b_1 = 0.45$, $b_2 = 1.0$, $\hat{\alpha}_1 = 1.2$ and $\hat{\alpha}_2 = 0.23$.

We will use a two-time scale perturbation method to solve this problem. The perturbation method is used because the equation (47) can not be solved analytical. In this method the solution is supposed to be a series of the eigenfunctions of the unperturbated problem(i.e. problem (47)-(50) with $\epsilon = 0$). We will only truncate to the first three eigenfunctions to construct an $O(\epsilon^3)$-approximation. Using this method we derive the following values

$$\begin{aligned} u_{max,dynamic,y} &= 12mm, \\ u_{max,dynamic,z} &= 2.2mm, \end{aligned}$$

for the maximum displacement of the tower in the y and z-direction respectively. So the maximum displacement of the tower is

$$u_{max,dynamic} = \sqrt{u^2_{max,dynamic,y} + u^2_{max,dynamic,z}} \cong 12mm. \tag{51}$$

This maximum displacement is at the top of the beam.

4. Conclusion

In this paper we considered the maximum dynamic displacement of the Old Church in Delft due to the swinging of the bell, the so-called Bourdon. We modelled the leaning Old Chuch by a skew vetical Euler-Bernoulli beam and we modelled the swinging of the Bourdon bell as a non-linear singular pendulum. First we obtain the values of the forces due to the bell acting on the Churh. We used these values to obtain that the maximum dynamic displacement of the Churh is $12mm$.

References

[1] Ferdinand Verhulst, *Nonlinear Differential Equations and Dynamical Systems*, (2th ed.), (Berlin: Springer Verlag, 1996).

[2] Jahnke-Ende, *Tafeln Höhern Funktionen*, (5th ed.), (Leipzig: B.G. Teubner, 1960).

[3] Maria L. Beconcini, Stefano Bennati, Walter Salvatore, *Structural characterisation of a medieval bell tower: First historical, experimental and numerical investigations*, (University of Pisa, Dept. of structural engineering, Pisa, Italy, 2001).