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CHAPTER 5

Hanging a Carillon in a Broek-system

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Abstract. A carillon is a musical instrument consisting of a (fairly large) set of bells that can be tolled by playing a keyboard, which is usually located one storey below the bells in a tower. Wires connect the keyboard to the clappers of the bells, forming an intricate web that is hinged to the walls. The web may vibrate, rub and tangle during play and some of the keys may require more pressure than others. The paper presents some methods to prevent these problems.

Keywords: geometric optimization, carillon, equilibration of musical instruments.

1. Introduction

A carillon is a musical instrument. It is a set of bells in a church tower that can be played by finger keys. The bells are hung high up in the tower. The keyboard, which is one metre long, is located one or more floors below. It is connected to the bells by an intricate web of wires, illustrated by figure 1.

There are two common ways to connect a key to a bell, the tumble system and the broek-system. This paper considers the broek-system only. As the tumble system is easier to design, it is nowadays the most common way to hang a carillon. However, for both playing and maintenance reasons, the broek-system is preferred. A sketch of the broek connection is shown in figure 2. In this particular example the wire has two segments that are connected by a ring, a so-called 'broekring', that is connected to the wall by a wire. By depressing the key the wire is pulled down and the clapper strikes the bell. Using this kind of wire construction *each and every* bell of the carillon needs to be connected to the keyboard. Figure 1 illustrates that this is not an easy task, especially not since there are quite a few requirements that an ideal broek-system has to meet. For one, the wires should not be too close to each other because they will swing a little during play.

The analysis of the system comprises the following stages.

1 We firstly study the statics of a single wire-bell connection. In particular we identify the force and displacement that needs to be applied to the keyboard wire to ring the bell, and see how this force depends upon the geometry of the wire configuration. A key feature of this calculation is identifying a suitable geometry of the broekring system which leads to an even force for the keyboard player when

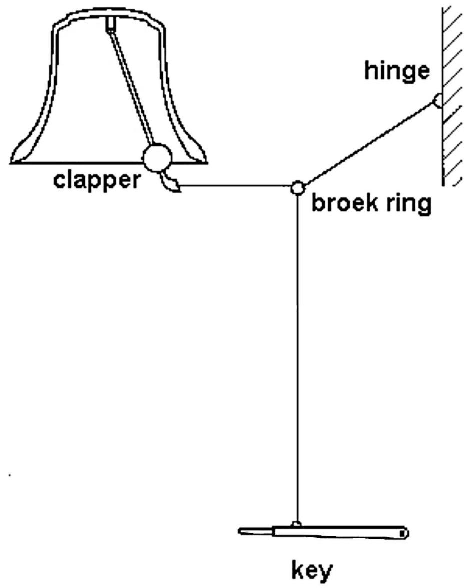

FIGURE 1. Schematic broekring connection

playing each bell.

2 We then use Lagrangian mechanics to study the (periodic) dynamics of a single wire-bell connection with one broekring. The oscillation of the system causes two problems. Firstly it makes it difficult to press the the same key too often. Secondly, the oscillations can potentially bring the wires for different bells into contact. Calculating the (frequency and) amplitude of these vibrations allow us to determine how far apart the wires should be.

3 Finally we apply the information determined in stages 1 and 2 above to a larger system with several bells. Each wire system can be hung in a certain geometry identified by 1 and the wires must be kept separate by 2. Applying each of these rules as constraints in the full network, we can identify an 'optimal' configuration by using a (partly stochastic) *greedy algorithm* to determine where the bells should be placed in the tower.

4 Throughout the text the reader finds four rules of thumb for the design of a carillon, that follow from our calculations.

2. Figures and facts

In this section we identify some key constraints of the geometry of the bells and of the wires connecting them to the keyboard.

2.1. Notation. The following symbols are used throughout the article.

N	=	number of bells
T	=	top of bell, the clapper is attached here.
C	=	end of the clapper, the ball is here
R	=	broekring
K	=	key
H	=	hinge where the broekring is attached to the wall
σ	=	angle of the clapper with the vertical
γ	=	angle between the clapper and the clapper-wire
α, β	=	angles at the broekring
l_0	=	length of the clapper
l_i	=	lengths of the other wires
m_c	=	mass of the clapper (varies from 0.3-30 kg (!))
m_i	=	masses of the wires
dC	=	distance the clapper moves (approx 2–5 cm)
dK	=	displacement of the key (approx 5 cm)

2.2. The general geometry. A carillon comprises N bells connected to an equal number of keys on the keyboard. N can be quite large and towers with 40 bells exist. In the tower we examined, the bells could be hung at one of two levels at a set of fixed points around the tower circumference. The position of the keyboard is fixed, as is the order of the keys, however there is freedom in the position of the bells around the towers. In Section 5 we will consider the problem of the optimal positioning of the *bells*. In this section we will assume that the bell is fixed and look at the resulting configuration of the *wires*. In the simplest case the keys are attached to the bells via a single broekring system, but in small towers containing many bells it is often necessary to use many broekrings. Whilst giving additional flexibility to the possible geometry of the wires, the use of more than one broekring is undesirable as it leads to significant extra friction in the system. We will look mostly at problems with one broekring, but will give a general formula for problems with many broekrings.

2.3. Playing the keyboard. The player sits at a horizontal, and straight, keyboard in which each key is depressed by dK, which is preferably the same for all keys. The wire attaching the key to the bell is initially vertical, with a broekring directly above the key. The motion of the wire causes the clapper of the bell to move a distance dC and then to strike the surface of the bell itself. When depressing a key to play the bell the player prefers the key to 'feel' the same when played, regardless of the bell to which it is attached. For all keys the player prefers a similar displacement dK **and** a similar force F needed to move the key. These preferences conflict, since FdK is equal to the gain of potential energy of the clapper and the clappers have varying weights. We come back to this in section 4.9.

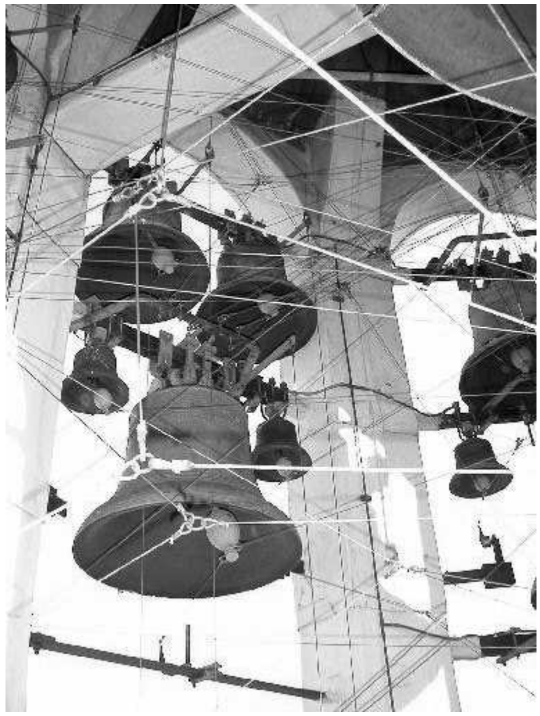

FIGURE 2. The carillon of the Zuiderkerk in Amsterdam

The displacement dK is roughly equal to 5 cm. The displacement dC obviously depends upon the size of the bell, and varies from 2-5 cm. Both distances are small compared to the length of the wires. The key is blocked from above to prevent it from moving upward since without this blocking the heavy clapper would move to its preferred, vertical position. If the clapper is *light* then a spring on the key returns the key to its rest position after it is depressed. The springs help somewhat to make the keys 'feel' the same while keeping dK fixed. In general, the only way to have similar feeling keys while keeping dK the same for all of them would be a system of springs at all keys, which would be hard to perfectly design.

2.4. The configuration of the wires. In a system with *one* broekring, the wire attaching the broekring to the clapper is called the *clapper-wire*, and we denote this as *wire 2* with length l_2 and mass m_2. The wire from the fixed point on the wall to the broekring is called the *broek-wire*, this we denote by *wire 3* with a corresponding length l_3 and mass m_3. Wire 1, the *key-wire*, goes down vertically from the broekring to the key and has length l_1 and mass m_1. In the usual configuration all of the wires are in the *same plane* by the rigidity of the wires. Only if there is some physical obstruction (e.g. a beam or a bell) this will not be the case.

The wire configuration is best described by the angles between the wires. At the broekring there are three angles $\angle(l_1, l_3), \angle(l_3, l_2), \angle(l_2, l_1)$ of which the third is the complement of the other two. We denoted $\alpha = \angle(l_1, l_3)$ and $\beta = \angle(l_3, l_2)$. The

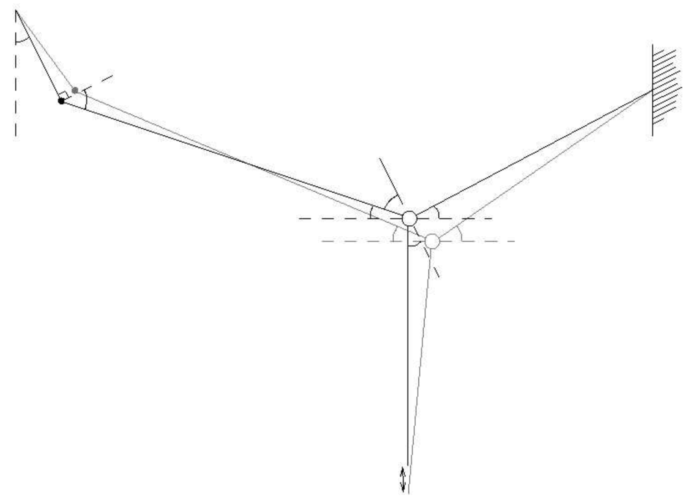

FIGURE 3. The one-broekring configuration in rest and after a vertical displacement dK. See figure 4 for the names of the angles between the wires.

angle between the clapper and the clapper wire is denoted by $\gamma = \angle(l_2, l_0)$. The angle between the clapper and the vertical at T is denoted by σ. If the key-wire is vertical then $\alpha + \beta + \gamma = 2\pi + \sigma$.

The order of which key is connected to which bell is (obviously) fixed, although the actual location of the bell is not. However, to work out the statics and dynamics of the above system we will assume that in this configuration the position of the bell is known, therefore, point T is known. Similarly, in a particular configuration we can fix the position of the keyboard and hence of the key K (to leading order). The angle of the clapper σ is known (to leading order) since we know the distance from the clapper to the bell and the shape of the bell. Given these constraints we need to determine the *unknown* position of the hinge H and of the broekring R. As the wires are in a plane, and we assume that the location of the bell and the key are known, this leads to a system with *two degrees of freedom* given by the heights of the hinge and the broekring.

3. The geometry of a single wire system

In this section we look further at the precise geometry of the one broekring system described above. We find simple equations for the changes in the angles of the wires when the key is depressed. We compute the ratio $|dC|: |dK|$, where dC represents the displacement of the clapper and dK represents the displacement of the key. We use the prefix d to emphasize that the equations are linearized, as discussed above in section 2.3. The ratio $|dC|: |dK|$ depends on the angles at the broekring R. By manipulating these angles the displacement and the 'feel' of the key can be adjusted. Again consider the one broekring system as in figure 2.

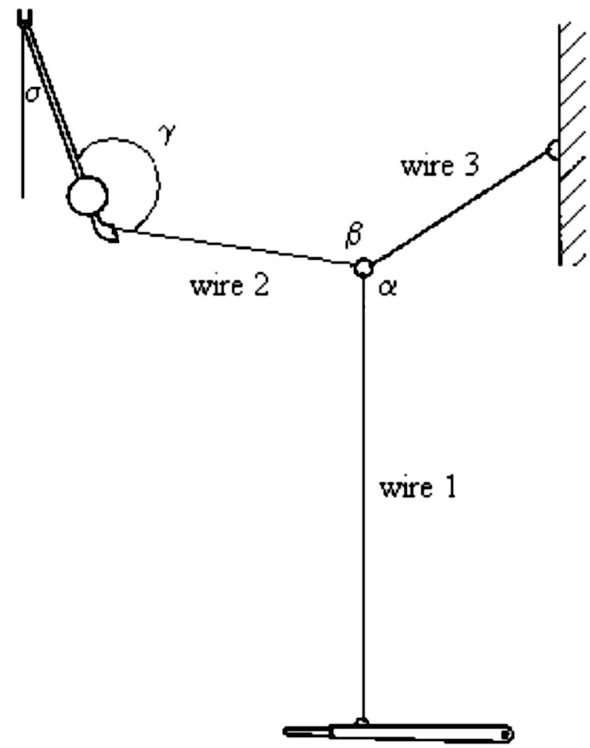

FIGURE 4. The angles in the wire system: the lengths of the wire 1, 2, and 3 are denoted by l_1, l_2, and l_3, respectively, and the length of the clapper is denoted by l_0.

The positions of H and T are fixed. The positions of R and C change when K is depressed. In the end, what we want to know is how the depression of K affects the position of C. As the broekring is small, we can neglect the sliding of the wire through the broekring R, so the lengths l_i of the wires are constant. This gives the four equations

$$\begin{aligned} |T-C| &= l_0 \\ |R-K| &= l_1 \\ |C-R| &= l_2 \\ |R-H| &= l_3 \end{aligned} \tag{1}$$

The unknowns are the positions of R and C. The depression of a key is a relatively minor movement compared to the length of the wire, so we may linearize the equations about the equilibrium position. The linearization of a distance $|A-B|$ is obtained from the inner product $\langle A-B, A-B\rangle = |A-B|^2$. We get four linear equations

$$\begin{array}{lcl} \langle T - C, dC \rangle & = & 0 \\ \langle R - K, dR - dK \rangle & = & 0 \\ \langle C - R, dC - dR \rangle & = & 0 \\ \langle R - H, dR \rangle & = & 0 \end{array} \tag{2}$$

If we assume that all points remain in a plane, then these are 4 linear equations with 4 unknowns, being the coordinates of R and C. The first equation and the last equation give the directions of dC and dR. The second and the third equation give the magnitudes of dC and dR. In particular, dC has an angle $\gamma \pm \pi/2$ with the clapper wire, while dR has an angle $\alpha \mp \pi/2$ with the key wire and an angle $\beta \pm \pi/2$ with the clapper wire. The third equation implies that $|dC| \colon |dR| = \sin \beta \colon \sin \gamma$. The depression of K is vertical, directed along $R - K$, so the second equation implies that $|dR| \colon |dK| = 1 \colon \sin \alpha$ and we find

$$\frac{|dC|}{|dK|} = \frac{\sin \beta}{\sin \gamma \sin \alpha} \tag{3}$$

In general the wire connection will not be in a plane and the coordinates of R and C comprise 6 unknowns. One can still compute R and C by geometric means, but it turns out that it is easier to use a force balance and that equation (3) remains valid even if the wire system is not in a plane. This we shall see in the section below.

In equation (3) we can influence the ratio $\frac{\sin \beta}{\sin \alpha}$ by moving H up or down. If H moves up then α increases and β decreases. Realistically $\pi/2 < \alpha, \beta < \pi$. So if H moves up the ratio $\frac{\sin \beta}{\sin \alpha}$ increases while it decreases if H moves down.

Rule of thumb. *If you want to increase the ratio $|dC| \colon |dK|$ then raise the hinge keeping the angle γ fixed or lower the position of the broekring while keeping the hinge fixed.*

Below we shall calculate the potential and kinetic energy of the wire system. For the potential energy the vertical components of the displacements are important, denoted by dR_2 and dC_2 (the displacement of the key is always vertical). The equations above imply that

$$|dR_2| = |dK| \text{ and } |dC_2| = \frac{\sin \sigma \sin \beta}{\sin \gamma \sin \alpha} |dK| \tag{4}$$

Note that this is only a linear approximation. In the real situation, $dR_2 \approx dK$, but it is slightly less by higher order ($O(dK^2)$) corrections. We will need second order approximations once we consider the vibrations of the system.

4. The periodic dynamics of a single wire system

As explained in Section 2, a carillon player prefers to play the carillon using approximately the same force for every key. The clapper weights vary from 0.3 to 30 kg, so there is a bit of a problem, even if we allow playing basses to be somewhat heavier than playing trebles. To understand which forces play an important role in the system, we first calculate the forces in a static system with massless wires. This can simply be done using Newton's laws. Incorporating the mass of the wires and the dynamics is done in a more general Lagrangian setting. For the computation

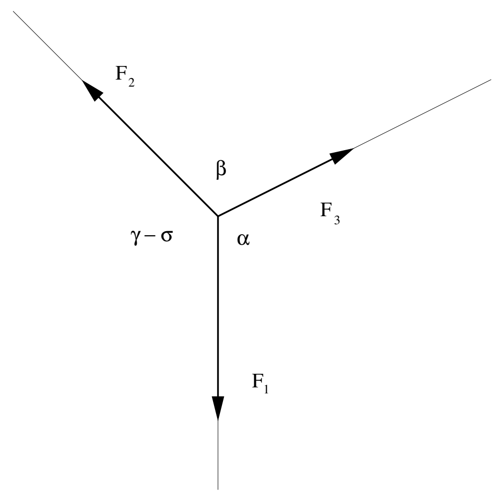

FIGURE 5. The force balance

of the forces and the energy we assume that the wires are rods, which is correct up to first order. For the computation of the vibrations, which is of second order, this assumption cannot be maintained.

4.1. Static forces for massless wires. We first calculate the force on the key if the clapper is in rest. This calculation generalizes equation (3) to wire systems with many broekrings. We assume that the mass m_c of the clapper is contained in C and that the wires are massless rods of constant length. A wire pulls at both its ends, being K, B, C or T, with forces that have equal magnitude but opposite sign by Newton's third law. We denote the force in the i-th wire by F_i.

At the broekring, the forces balance by Newton's first law as shown in figure 4.1, so

$$\frac{|F_1|}{|F_3|} = \frac{\sin\beta}{\sin\alpha} \tag{5}$$

At the clapper the balance of forces gives

$$\frac{|F_3|}{|m_c g|} = \frac{\sin\sigma}{\sin\gamma} \tag{6}$$

Since $|F_1|$ is the force on the key, we find that the clapper exerts a force on the key of

$$m_c g \frac{\sin\beta\sin\sigma}{\sin\alpha\sin\gamma}. \tag{7}$$

This force becomes singular in the cases $\sin\alpha = 0$ or $\sin\gamma = 0$. Exactly the same singular case occurs when the wires do have mass, see also figure 7 below.

So far we carried out the calculations for wire connections that have one broekring only. It is possible to extend the calculations to systems that have many broekrings, as in figure 6.

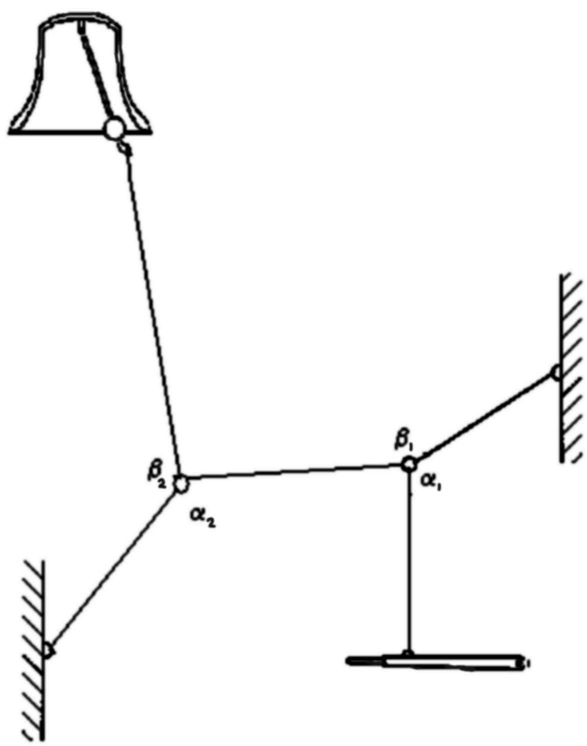

FIGURE 6. Two broekrings

4.2. Configuration with more broekrings. In a more general configuration, the wire connects the key to the bell by k broekrings R_i, $i = 1, \dots, k$. Following exactly the same procedure as above, we obtain $2(k+1)$ conditions from linearising the fixed distances $|T - C|$, $|C - R_k|$, $|R_i - H_i|$ and $|R_i - R_{i-1}|$, where $R_0 = K$ and the positions of C and the broekrings R_i are unknown. If we assume that, as before, all the points remain in one plane there are $2(k+1)$ equations with $2(k+1)$ unknowns (the coordinates of C and the R_i). In this case, dR_i can be computed from dR_{i-1}. Let α_i be the angle between the wire connecting R_i to R_{i-1} and the wire connecting R_i to H_i. And, let β_i be the angle between the wire connecting R_i to R_{i+1} and the wire connecting R_i to H_i. The angles α_i and β_i are shown in figure 6 for two broekrings. Then the relation between the distances dR_i and dR_{i-1} over which the broekrings move, can be obtained from the above equations in a similar way as for the one broekring system, and it is given by $|dR_i|/|dR_{i-1}| = \sin(\beta_{i-1})/\sin(\alpha_i)$. We introduce the notation

$$w_i = \frac{\sin(\beta_{i-1})}{\sin(\alpha_i)} \tag{8}$$

and call this the *index* at the i-th broekring. For a system with k broekrings, the ratio $|dC|\colon|dK|$ is the product over all the indices with an additional contribution of the angles at the clapper, according to the following formula

$$\begin{aligned}\frac{|dC|}{|dK|} &= \frac{|dC|}{|dR_k|}\cdot\frac{|dR_k|}{|dR_{k-1}|}\cdots\frac{|dR_2|}{|dR_1|}\cdot\frac{|dR_1|}{|dK|} \\ &= \frac{1}{\sin(\alpha_1)}\prod_{i=1}^{k} w_i \qquad (9)\end{aligned}$$

where we assume that the displacement of the key $dK = dR_0$ is vertical. In the simple configuration with one broekring this again reduces to

$$\frac{|dC|}{|dK|} = \frac{\sin(\beta)}{\sin(\gamma)\sin(\alpha)}. \qquad (10)$$

As mentioned in Section 2, the fraction $|dC|\colon|dK|$ is important for the design of the carillon and again we can influence it by moving the hinges

Rule of thumb. *If you want to increase the ratio $|dC|\colon|dK|$ then raise the hinge of the first broekring keeping the angle γ fixed or lower the position of the first broekring keeping the hinge fixed.*

4.3. The Lagrangian of the system with massive wires. To find an expression for the force that is needed to play a key in case the masses of the wires are relatively large, we calculate the total energy for a single bell system, and the change in potential and kinetic energy under a small vertical displacement dK of the key.

The natural way to describe this mechanical system is by computing its Lagrangian. In order to obtain the total energy, we first determine the potential energy and the kinetic energy in every single point in the system and then integrate over the wires. We shall need second order approximations.

4.4. Potential energy. For the computation of the potential energy of the whole system, it is important to know where the gravitational force on each wire and on the clapper works, so where the center of mass is located. As the clapper is a rather thin pole with a large massive ball at the hanging end, we assume that all the mass of the clapper is located in this end. The wires are more or less uniform, so we can take their centers of mass in the middle of each wire. Suppose that the coordinates of the end points of a uniform wire are $\underline{a} = (a_1, a_2)$ and $\underline{b} = (b_1, b_2)$. Then the potential energy of the wire is $mg(a_2 + b_2)/2$, where m denotes its mass. This applies to wires 2 and 3. We treat wire 1 in a different way, since both the weight of this wire and the force on the key approximately work along the same line in the vertical direction. The force on the key is always present: in rest, the key is blocked from above to prevent it from moving upward since without this blocking the rather heavy clapper would move to its preferred, vertical position. (Recall that the distance of the clapper to the bell in rest should be only 2 - 5 cm., so $\sigma > 0$.) When a bell is played, this force should first be overcome and then some additional force should be applied to move the clapper. We simply put this force and the gravitational force on the wire into one 'tension' τ, which in fact consists of the tension in rest (directed upwards and canceled by the blocking force) and the (downward) tension added by the player.

The potential energy depends on the end points K, R, C, H of the wires. Denote the coordinates of K, R, H, C in the initial position by $(K_1, K_2), (R_1, R_2), (H_1, H_2)$ and (C_1, C_2) respectively. Under the assumption of uniform mass distribution, the potential energy of the system in equilibrium is given by

$$V_0 = C_2(m_c + \frac{m_2}{2})g + R_2\frac{(m_2+m_3)g}{2} + H_2\frac{m_3 g}{2} + K_2\tau$$

Let $V(dK)$ be the potential energy of the system, with the key in vertical position $K_2 + dK$, such that $V(0) = 0$. (This means that V(dK) is the relative potential energy with respect to equilibrium.) Under the above assumptions on the centers of mass the potential energy of the system is given by

$$V(dK) = dC_2(m_c + \frac{m_2}{2})g + dR_2\frac{(m_2+m_3)g}{2} + dK\tau$$

In equation (4) we find the linear dependence of dC_2 and dR_2 on dK. The linear part of the Lagrangian determines the equilibrium. We also need the quadratic part once we consider the vibrations of the system. Introducing the exact angles $\alpha + d\alpha$, $\beta + d\beta$, $\gamma + d\gamma$ and $\sigma + d\sigma$ after displacement, this quadratic part can be computed using geometrical constraints and Taylor expansions. We found that

$$\begin{aligned} V(dK) &= dK\left((m_c + \tfrac{m_2}{2})g\tfrac{\sin\sigma\sin\beta}{\sin\gamma\sin\alpha} + \tfrac{(m_2+m_3)g}{2} + \tau\right) \\ &+ \tfrac{dK^2}{2\sin^2\alpha}\cdot\left(m_c g\tfrac{\cos\sigma\sin^2\beta}{l_0\sin^2\gamma} + m_2 g\tfrac{\cos(\sigma-\gamma)\sin^2(\sigma-\alpha)}{2l_2\sin^2\gamma} - \tfrac{m_3 g\cos\alpha}{2l_3}\right) \\ &+ O(dK^3) \end{aligned}$$

Note that, since RH is attached to the wall, the potential energy involving m_3 must equal the potential energy for a simple rod hinged at H. This is indeed the case.

4.5. Kinetic energy. Consider a rod of uniformly distributed mass m that is moving. Its movement can be described by the coordinates of its end points $\underline{a}, \underline{b}$ depending on time. If we denote the derivative with respect to time of the coordinates a and b by $\dot{a}, \dot{b}$, then the kinetic energy of the rod is

$$\frac{1}{2}m\int_{t=0}^{1}|t\underline{\dot{a}} + (1-t)\underline{\dot{b}}|^2 dt = \frac{m(|\underline{\dot{a}}|^2 + \langle\underline{\dot{a}},\underline{\dot{b}}\rangle + |\underline{\dot{b}}|^2)}{6}.$$

So the kinetic energy of the broek-system is given by

$$\frac{m_1(|\dot{K}|^2 + \langle\dot{K},\dot{R}\rangle + |\dot{R}|^2)}{6} + \frac{m_2(|\dot{R}|^2 + \langle\dot{R},\dot{C}\rangle + |\dot{C}|^2)}{6} + \frac{m_3|\dot{R}|^2}{6} + \frac{m_c|\dot{C}|^2}{2}.$$

In section 3 we computed the relations between dK, dR and dC and we found that

$$\begin{aligned} |dR| &= |dK|/\sin\alpha \quad , \quad \angle(dR, dK) = \alpha - \tfrac{\pi}{2} \\ |dC| &= |dK|\tfrac{\sin\beta}{\sin\alpha\sin\gamma} \quad , \quad \angle(dR, dC) = \beta + \gamma \end{aligned}$$

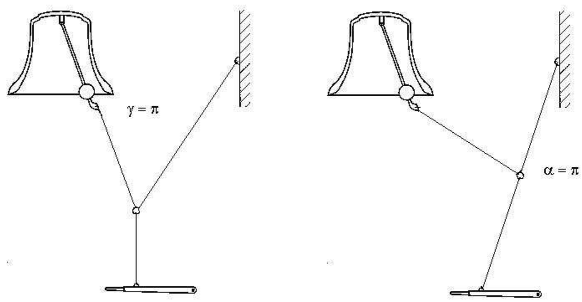

FIGURE 7. Two singular cases. When the key-wire is vertical, the broek-wire and key-wire hang along the wall in the right hand case

Denote the ratio $|dR|/|dK|$ by ρ_R and the ratio $|dC|/|dK|$ by ρ_C. Then using relations as $\dot{R} = \dot{K}dR/dK$, the kinetic energy in terms of $\dot{K}$ is

$$T(K) = \frac{1}{6}|\dot{K}|^2 \left(m_1(1+\rho_R \sin\alpha + \rho_R^2) + m_2(\rho_R^2 + \cos(\beta+\gamma)\rho_R\rho_C + \rho_C^2) + \right.$$
$$\left. + m_3{\rho_R}^2 + 3m_c\rho_C^2\right). \tag{11}$$

4.6. Lagrangian motion. With the obtained T and V we can write down the Lagrangian $\mathcal{L} = T - V$ of the system, and hence the equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{K}} - \frac{\partial \mathcal{L}}{\partial K} = 0.$$

Here $\frac{\partial \mathcal{L}}{\partial K}$ can be seen as the force in the system, which derives in this case from a potential. In equilibrium, the force acting on the system should be equal to zero, meaning that $\frac{\partial \mathcal{L}}{\partial K} = -\frac{\mathrm{d}V}{\mathrm{d}K} = 0$ in equilibrium. This immediately gives a relation for τ, since it means that the coefficient of the linear part of V should be zero. The $\mathcal{O}(dK^2)$ terms in V have to balance with T. From this we can derive natural frequencies of the system, that can give a clue about the amount of vibration one can expect in the wires.

We will do these calculations in a system with massless wires. In this case the system should more or less satisfy the 'classical' intuition; therefore it is a good check for our calculations.

4.7. Dynamics of massless wires. In case of massless wires only terms concerning the clapper remain. From $\frac{\mathrm{d}V}{\mathrm{d}K} = 0$ we derive in this case

$$\tau = m_c g \frac{\sin\sigma \sin\beta}{\sin\gamma \sin\alpha} \tag{12}$$

which is exactly the same expression as (7). This becomes singular in the cases $\sin\alpha = 0$ or $\sin\gamma = 0$. Exactly the same singular cases appear in the full expression for τ when the wires do have mass. From the sketches 7 it is immediately clear that

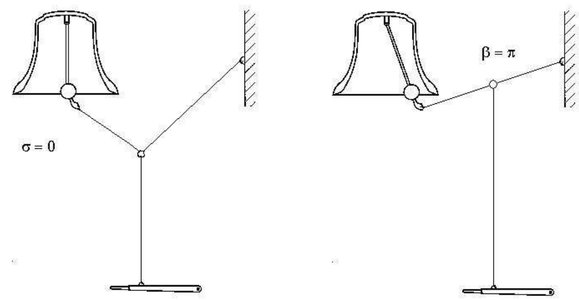

FIGURE 8. A massless system that does not exert a force on the key

it should indeed cost an infinite amount of work to move the key or clapper when the configuration satisfies either case.

In this massless case it is also easy to understand how the system should behave when there is no tension at all, so when the key is not blocked or played. The system then just hangs, and one would expect that the clapper would hang in its preferred, vertical position or would somehow not be able to reach that position. This indeed follows from the assumption $\tau = 0$: the solutions are $\sin\sigma = 0$ and $\sin\beta = 0$, leading to the configurations in figure 8.

4.8. Swinging and vibration. In reality a wire is not a rod but it is an elastic string. The key strokes induce displacements of the broekrings, causing vibrations. This vibration should not be too much, since the wires are, in an ideal situation, not allowed to touch each other. To get an idea about the possible vibration, we derive the natural frequencies of the system from the Hamiltonian $H = T + V$, These natural frequencies tell us what happens if you *twang* a wire.

For massless wires, putting $H = 0$ and comparing equations 4.4 and 11 yields the differential equation

$$\dot{K}^2 = \frac{g\cos(\sigma)}{l_0} dK^2.$$

Recall that the coefficient of the linear term in V is zero. If we put $x = e^{i\omega t}$, this results in

$$\omega = 2\pi f = \sqrt{\frac{g\cos(\sigma)}{l_0}}.$$

Here f is the frequency. For typical values of the parameters, say $l_0 = 0.5$, $\cos\sigma = 0.8$, this results with $g = 10$ in $\omega = 4$, or a frequency of about 0.6 Hz. From our observations in the church we estimated a frequency of 0.2 Hz.

The same kind of calculations can be done for wires with mass, for which one ends up with more complicated expression that we do not display here.

When playing the carillon, the vertical wire will always move a bit. But since the vertical wires are all attached to the keyboard, they are very close to each other, even when they are in rest. This means that a small sidewards movement can already cause touching wires, which we should prevent.

It is difficult to give a precise bound on the amplitude, since this will depend on the geometry, the tension and the number of key strokes during play. However, we observed that the maximal amplitude of the vibration occurs at the first broekring R_1 at the end of the vertical first wire segment. Its displacement twangs the wire, as represented by the horizontal component of dR_1. When the vertical movement of the wire is dK, the sideways swing of the wire at the position R_1 is approximately $dR_1 = dK/\tan\alpha$, which means that the configuration should optimally satisfy $\alpha = \pi/2$.

Rule of thumb. *The vibration of the first broekring can be damped by putting the hinge at the same level as the ring.*

4.9. Playing the carillon. We have already addressed some rules of thumb for the angles in the system, that follow from conditions on the swing when playing the carillon. However, we did not mention the requirement that is maybe the most important one: the keyboard player roughly wants every key to 'feel' the same. The basses may need some more force to play than the trebles, but the amount of force that is needed to play neighbouring tones should not differ much. We suppose that the travel dK is constant for all keys, i.e., that the difference between the upper and the lower position is the same for each key. The question now is: under what conditions is the tension that the player should add to play a bell (about) the same for each key?

A first idea can again be obtained from the system with massless wires. We assume that there is a weight W that 'helps' the player to move the key down (which is in our former calculations incorporated in τ). This weight may contain the mass of the wires, that we assume to be concentrated at the key, if not negligible. Since the total energy should be remained, this means that the following should be satisfied:

Energy input by player = potential energy gain of clapper - energy lost by help.

Note, that the clapper moves upward when the key is played, while the helping weight moves downward. This means that the clapper gains potential energy, while the weight loses potential energy. In a formula we have

$$\tau_{\text{player}} dK = m_c g l_0 \sin\sigma \cdot d\sigma - W dK. \tag{13}$$

If each key should need the same amount of added tension, this means that $\tau_{\text{player}} dK$ should be constant. In other words, for $W = 0$ we obtain the restriction

$$m_c l_0 \sin\sigma \cdot d\sigma = C, \tag{14}$$

where C is the same constant for all bells!

However, detailed information about the St. John's tower in Gouda tells us that this requirement is impossible to satisfy. There the specification of the bells gives:

large bell $l_0 \sin\sigma \approx 50$ cm, $m_c \approx 30$ kg
small bell $l_0 \sin\sigma \approx 10$ cm, $m_c \approx 0.3$ kg.

To impose (14) for this tower, the angles of swing in the smallest and the largest bells should satisfy

$$d\sigma_{\text{large}} = 50 \cdot d\sigma_{\text{small}},$$

which is impossible (it would for instance mean that a small bell would move by 1 degree and a large one by 50 degrees). This actually tells us, that the helping weight W is really needed at the larger bells to obtain an evenly played keyboard.

These calculations nicely correspond to our observations. In a real carillon system the wires are always very long and quite thick, and have a total mass of up to 1 kg. This means that the mass is not negligible for small bells with clapper weights of only 0.3 kg. The mass of the wires that we put in W may even be so large compared to m_c, that the right hand side of (13) is negative. This would mean that help is needed to move the clapper back to its original position, which is indeed the case in real carillons. We observed that some of the smaller clappers had a spring to move them back after they had been played.

On the other hand, we observed springs that helped to move the keys of larger bells, which could account for the extra weight W needed in order to obtain an evenly played keyboard.

5. The 3D configuration of the broek-system

There are two interesting optimization problems related to the placement of bells in a tower. In the first problem we view the tower from above and we try, mainly to get an idea of the situation, to find a configuration such that the intersection points (of the projections) of the wires are as far apart from each other as possible. Of course there is no guarantee that this configuration is the best in the three dimensional case.

In the second optimization problem we try to approximate the three-dimensional real situation in a belfry. Then we would like to maximize the smallest distance between all the wires so that they're as far apart from each other as possible to avoid that they will touch when the carillon is played.

5.1. The two-dimensional problem. Given the possible positions of the bells ($\{b_i \in \mathbb{R}^2 \mid i = 1, 2, \ldots, n\}$) on the circumference of the tower and the position of the attachment points to the keys ($\{k_i \in \mathbb{R}^2 \mid i = 1, 2, \ldots, n\}$) in the keyboard, the place of each wire, being a straight line through these two points, is uniquely determined. If we start with n keys and n bells, then the wires will intersect in (at most) $\frac{1}{2}(n-1)n$ points. Our aim is to design the configuration of the bells in such a way that these intersection points lie as far apart as possible. Therefore, we first look at the distances between all these points. There are

$$\binom{\frac{1}{2}(n-1)n}{2}$$

distances between the intersection points which need to be computed. Among these distances we have to find the smallest distance and then the optimization problem is to decide which positioning of the bells maximizes this minimal distance. A

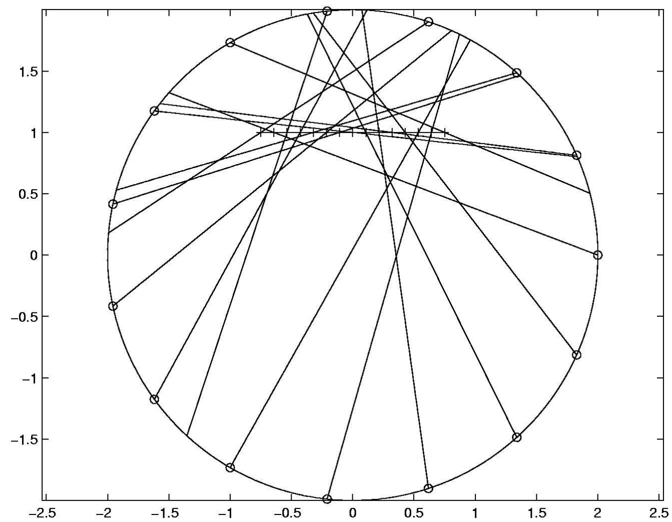

FIGURE 9. A belfry with a radius of 2 m with 15 bells seen from above. The distance between the intersections is never less than 1.31 cm.

computer program which tries different permutations of the bells randomly can approximate the maximum. However, this is a quite cumbersome task since for n bells there are $n!$ configurations that need to be checked. Thus for one level in a belfry with 15 bells, it will take $0.714 \cdot 10^{16}$ computations, and for a belfry with 45 bells the number of computations will be $0.586 \cdot 10^{62}$.

Computations with Matlab yield for a belfry with 15 bells and a radius of 2 m after a 698 tries, a configuration where each distance between two intersections is greater than 1.3 cm (see figure 9).

Notice that there are two more general interesting geometrical questions connected to this problem. The first question, important to obtain a good estimate in the optimization problem, is the following. How should n points be placed in a unit disk, such that the smallest distance between two points is maximal? For $n = 2, 3, 4, 5, 6$, this distance is $2, \sqrt{3}, \sqrt{2}, \sqrt{(5-\sqrt{5})/2}, 1$, respectively, and the points have to be placed on the boundary. For $n = 7$ the maximal distance is also 1, but one point has to be placed in the interior of the disk. But what about higher n?

The second question is the following: consider two sets of n colored points in the real plane, $\{b_i \in \mathbb{R}^2 \mid i = 1, 2, \ldots, n\}$ (blue, the keyboard) and $\{k_i \in \mathbb{R}^2 \mid i = 1, 2, \ldots, n\}$ (black, the bells). How can we connect each of the blue points with a different black point with a straight line such that the distance between the intersection points is as large as possible.

5.2. The three-dimensional problem. Let us now consider the belfry in three dimensions. So, assume we can place bells in an equidistant way on the

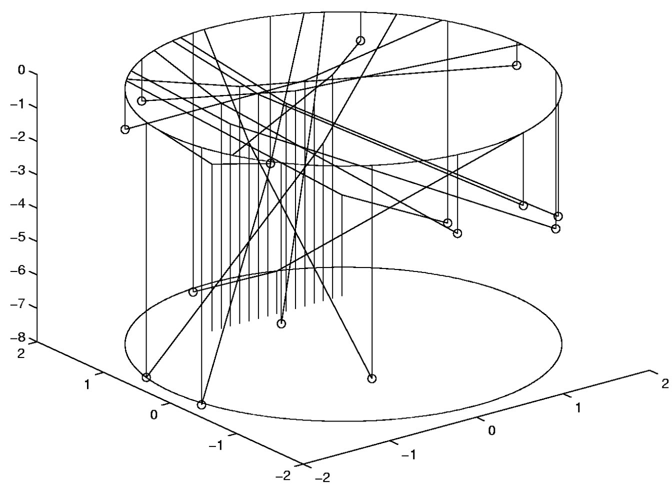

FIGURE 10. A belfry with a radius of 2 m with 15 bells. The distance between the wires is greater than 2.95 cm.

circumference of the belfry, where the height is free to choose. Also, we assume that the wires which connect the broekring to the wall are all attached to the wall at a fixed height on the circumference of the belfry. Then we find that once the horizontal position of the bell is chosen, the vertical position of the bell is completely determined by the formulas

$$\begin{cases} \frac{\sin\beta}{\sin\gamma\sin\alpha} = \frac{|dC|}{|dK|} \\ \alpha+\beta+\gamma-2\pi=\sigma \end{cases}$$

because $|dC|/|dK|$ and σ are fixed (see figure 4). Starting with an arbitrary permutation of the bells, we compute the height of each bell by the above formulas, and then we compute all the distances between the pieces of wire. We would again like the smallest of these distances to be as large as possible. It is astonishing that it is hard to find good configurations by a random permutation of the bells. After 1112 permutations of 15 bells in a belfry with a radius of 2 meters we find that for the optimal configuration the smallest distance is 2.95 cm (see figure 10). The geometrical character of this problem is similar tot the two-dimensional case.

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